

SYSTEMS OF DIAGRAM CATEGORIES AND K-THEORY. I

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ABSTRACT. To any left system of diagram categories or to any left pointed dérivateur a K -theory space is associated. This K -theory space is shown to be canonically an infinite loop space and to have a lot of common properties with Waldhausen's K -theory. A weaker version of additivity is shown. Also, Quillen's K -theory of a large class of exact categories including the abelian categories is proved to be a retract of the K -theory of the associated dérivateur.

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INTRODUCTION

The object of this paper is to construct the Waldhausen K -theory for systems of diagram categories and pointed dérivateurs respectively. The general formalism for them was developed by Heller [9], Grothendieck [8], and Franke [6].

A system of diagram categories (respectively dérivateur) is a hyperfunctor

$$\mathbf{B} : \mathcal{D}ia^{\text{op}} \longrightarrow \mathbf{CAT}$$

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defined on a 2-subcategory of the 2-category of small categories **Cat** (we shall refer to *Dia* as a *category of diagrams*) and taking values in the “large” 2-category of categories **CAT**. We also require *Dia* to contain the 2-category of posets (=finite partially ordered sets) *Ord*. A typical example of such a hyperfunctor is given by the map

$$I \in \text{Dia} \longmapsto \text{Ho } \mathcal{C}^I,$$

where \mathcal{C} is a closed model category, $\text{Ho } \mathcal{C}^I$ is the homotopy category of the functor category \mathcal{C}^I (the structure of a closed model category on \mathcal{C}^I is naturally defined for appropriate diagrams).

First we define the *S*.-construction for an appropriate **B** and then its *K*-theory space $K(\mathbf{B})$ as the loop space of $|i.S.\mathbf{B}|$, where $iS_n\mathbf{B}$ stands for the subcategory of isomorphisms in each category $S_n\mathbf{B}$, $n \geq 0$. The space $K(\mathbf{B})$ is canonically an infinite loop space by Segal’s machine [19]. The additivity theorem is discussed for this *K*-theory. Developing Waldhausen’s machinery [26] for objects in question the additivity theorem implies that one can also think of $K(\mathbf{B})$ in terms of the following connective Ω -spectrum. Namely, it is given by the sequence of spaces

$$\Omega|i.S.\mathbf{B}|, \Omega|i.S.S.\mathbf{B}|, \dots, \Omega|i.S.^n\mathbf{B}|, \dots$$

where the multisimplicial objects $i.S.^n\mathbf{B}$, $n \geq 1$, are obtained by iterating the *S*.-construction. Though the additivity theorem remains open in the general case (see also [14, Conjecture 3]), a weaker version does hold.

THEOREM. *The additivity theorem is valid for the space*

$$\Omega^\infty|i.S.^\infty\mathbf{B}| = \lim_n \Omega^n|i.S.^n\mathbf{B}|$$

excluding pathological cases we never have in practice.

The strong form of additivity is shown in [7] for complicial dérivateurs.

We also remark that the Grothendieck group $K_0(\mathcal{E})$ of an exact category \mathcal{E} is naturally isomorphic to the group $K_0(\mathbf{D}^b(\mathcal{E}))$ of the associated dérivateur $\mathbf{D}^b(\mathcal{E})$ given by the hyperfunctor

$$I \longmapsto D^b(\mathcal{E}^I)$$

where $D^b(\mathcal{E}^I)$ is the derived category for the exact functor category \mathcal{E}^I .

One can obtain some relation between Quillen’s *K*-theory $K(\mathcal{E})$ and $K(\mathbf{D}^b(\mathcal{E}))$ for a large class of exact categories including the abelian categories.

THEOREM. *Let \mathcal{E} be an extension closed full exact subcategory of an abelian category \mathcal{A} satisfying the conditions of the Resolution Theorem. That is*

(1) *if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is exact in \mathcal{A} and $M, M'' \in \mathcal{E}$, then $M' \in \mathcal{E}$ and*

(2) *for any object $M \in \mathcal{A}$ there is a finite resolution $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ with $P_i \in \mathcal{E}$.*

Then a natural map

$$K(\rho) : K(\mathcal{E}) \longrightarrow K(\mathbf{D}^b(\mathcal{E}))$$

is a split inclusion in homotopy. There is a map

$$p : K(\mathbf{D}^b(\mathcal{E})) \longrightarrow K(\mathcal{E})$$

which is left inverse to it. That is $p \circ K(\rho)$ is homotopic to the identity. In particular, each K -group $K_n(\mathcal{E})$ is a direct summand of $K_n(\mathbf{D}^b(\mathcal{E}))$.

The problem whether Quillen's K -theory $K(\mathcal{E})$ can be reconstructed from $\mathbf{D}^b(\mathcal{E})$ ("the first Maltsiniotis conjecture") remains open.

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1. SYSTEMS OF DIAGRAM CATEGORIES

In this section the reader will be attacked by categorical formalities and definitions of concepts. A lot of analogous statements for this section are also stated in [6]. Since the latter work contains a lot of slight errata and this paper is one of the first in this direction, we prove them once more in details *in order to be sure* that nothing goes wrong. This makes therefore our paper self-contained. We follow here the original terminology of Franke [6]. The author must admit that the formal portion of this section is maximally shortened.

1.1. Notations. Let I be a category. For a subcategory J of I and $x \in I$, we shall denote by J/x the following comma category. Objects are the pairs (y, φ) where $y \in J$ and $\varphi : y \rightarrow x$ is a morphism in I . Morphisms from (y, φ) to (y', φ') are given by morphisms $\psi : y \rightarrow y'$ in J such that $\varphi = \varphi' \psi$. The category $J \setminus x$ consists of pairs (y, φ) with $y \in J$ and $\varphi : x \rightarrow y$. Morphisms are defined similar to those of J/x . If $K \subseteq \text{Ob } I$ is a subclass of objects, we shall denote by $I - K$ the full subcategory of I with the class of objects $I - K$. In particular, if $K = \{x\}$ has just one object, we shall also denote this subcategory by $I - x$. If $f : J \rightarrow I$ is a functor, the categories f/x and $f \setminus x$ have objects $(y \in J, \varphi : f(y) \rightarrow x)$ and $(y \in J, \varphi : x \rightarrow f(y))$. If f is the inclusion of a subcategory, this is the same as J/x and $J \setminus x$.

Given a non-negative integer n , by Δ^n denote the totally ordered set $\{0 < 1 < \dots < n\}$. For $i \leq n+1$, the map $d_i : \Delta^n \rightarrow \Delta^{n+1}$ is the monotonic injection not containing i in its image and $s_i : \Delta^n \rightarrow \Delta^{n-1}$ is the monotonic surjection satisfying $s_i(i) = s_i(i+1)$.

1.2. The axioms. For the notions of the 2-category and 2-functor we refer the reader to [12]. In what follows we use the term "poset" as an abbreviation of "finite partially ordered set". Every poset can be considered as a category in which $\text{Hom}(x, y)$ has precisely one element $x \leq y$, and is empty otherwise. The 2-category of the posets (respectively the finite categories without cycles) we shall denote by Ord (respectively by Dirf).

Let Dia be a full 2-subcategory of the 2-category \mathbf{Cat} of small categories that contains the 2-category Ord . In what follows we assume that Dia satisfies the following conditions:

- (1) Dia is closed under finite sums and finite products;
- (2) for any functor $f : I \rightarrow J$ in Dia and for any object y of J , the categories f/y and $f \setminus y$ are in Dia .

We shall also refer to \mathcal{Dia} as a *category of diagrams*.

Given $I \in \mathcal{Dia}$, let I^* be I with an initial and final object \star added. For any x and y in I , there is the unique morphism from x to y in I^* which factorizes through \star . We shall refer to this morphism as the zero morphism. If $I \in \mathcal{Ord}$ and $x \leq y$ there is one more morphism from x to y in I^* , and there are no other morphisms. The composition is defined in the obvious way. Let \mathcal{Dia}^* be a 2-subcategory of the 2-category \mathbf{Cat} whose objects are those of \mathcal{Dia} and whose horizontal morphisms $I \longrightarrow J$ are given by functors $I^* \longrightarrow J^*$ mapping \star to \star , and let bimorphisms be natural transformations between functors from I^* to J^* .

A *presystem of diagram categories of the domain \mathcal{Dia}* or just a *presystem of diagram categories* is a functor

$$(1) \quad \mathbf{C} : \mathcal{Dia}^{\star\text{op}} \longrightarrow \mathbf{CAT}$$

from \mathcal{Dia}^* to the category \mathbf{CAT} of categories (not necessarily small) satisfying the Functoriality Axiom below. So to each category I in \mathcal{Dia}^* there is associated a category \mathbf{C}_I , and to each map $f : I \longrightarrow J$ in \mathcal{Dia}^* a functor $f^* = \mathbf{C}(f) : \mathbf{C}_J \longrightarrow \mathbf{C}_I$.

Functoriality Axiom. The following conditions hold:

- ◇ to each natural transformation $\varphi : f \longrightarrow g$ a natural transformation $\varphi^* : f^* \longrightarrow g^*$ is associated and the maps $f \longrightarrow f^*$ and $\varphi \longrightarrow \varphi^*$ define a functor from $\text{Hom}(I, J)$ to the category of functors from \mathbf{C}_J to \mathbf{C}_I ;
- ◇ if

$$K \xrightarrow{f} I \begin{matrix} \xrightarrow{g} \\ \xleftarrow{g'} \end{matrix} J \xrightarrow{h} L$$

are morphisms and $\varphi : g \longrightarrow g'$ is a bimorphism, then $f^* \circ \varphi^* = (\varphi \circ f)^*$ and $\varphi^* \circ h^* = (h \circ \varphi)^*$.

From now on let us fix a category of diagrams \mathcal{Dia} . To any category \mathcal{C} one associates a presystem of diagram categories which takes a category I of \mathcal{Dia}^* to the functor category

$$\mathcal{C}^{I^*} = \text{Hom}(I^*, \mathcal{C})$$

and a map $f : I \longrightarrow J$ to the map

$$f^* : \mathcal{C}^{J^*} \longrightarrow \mathcal{C}^{I^*}, \quad X \longmapsto X \circ f.$$

A *morphism* $F : \mathbf{C} \longrightarrow \mathbf{C}'$ between two presystems of diagram categories \mathbf{C} and \mathbf{C}' consists of the following data:

- (1) for any $I \in \mathcal{Dia}^*$, a functor $F : \mathbf{C}_I \longrightarrow \mathbf{C}'_I$;
- (2) for any map $f : I \longrightarrow J$ in \mathcal{Dia}^* , an isomorphism of functors $\iota_{F,f} : f^* F \xrightarrow{\sim} F f^*$.

We also assume the following conditions to hold for $\iota_{F,f}$:

- ◇ for any $I \in \mathcal{Dia}^*$, $\iota_{F,1_I} = 1_F$;

◇ for any two composable maps $I \xrightarrow{f} J \xrightarrow{g} K$ in \mathcal{Dia}^* , the diagram

$$\begin{array}{ccc} f^* g^* F & \xrightarrow{\iota_{F, gf}} & F f^* g^* \\ & \searrow f^* \iota_{F, g} \quad \nearrow \iota_{F, f g^*} & \\ & f^* F g^* & \end{array}$$

is commutative;

◇ for any bimorphism $\varphi : f \longrightarrow g$ in \mathcal{Dia}^* , we have the following commutative square.

$$\begin{array}{ccc} f^* F & \xrightarrow{\iota_{F, f}} & F f^* \\ \varphi^* F \downarrow & & \downarrow F \varphi^* \\ g^* F & \xrightarrow{\iota_{F, g}} & F g^* \end{array}$$

A morphism $F : \mathbf{C} \longrightarrow \mathbf{C}'$ is an *equivalence* if for any $I \in \mathcal{Dia}^*$ the functor $F : \mathbf{C}_I \longrightarrow \mathbf{C}'_I$ is an equivalence of categories.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be presystems of diagram categories. By the *fibred product* of a pair of morphisms $F : \mathbf{A} \longrightarrow \mathbf{C}$ and $G : \mathbf{B} \longrightarrow \mathbf{C}$ is meant the following data:

◇ for any $I \in \mathcal{Dia}^*$, the category $\prod(F, G)_I$ whose objects are the triples

$$(A, c, B), \quad A \in \mathbf{A}_I, \quad B \in \mathbf{B}_I, \quad c : F(A) \xrightarrow{\sim} G(B),$$

and where a morphism from (A, c, B) to (A', c', B') is a pair of morphisms (a, b) compatible with the isomorphisms c and c' ;

◇ for any map $f : I \longrightarrow J$ in \mathcal{Dia}^* , the functor

$$f^* = f^*_{\prod(F, G)} : \prod(F, G)_J \longrightarrow \prod(F, G)_I$$

defined by

$$(A, c, B) \longmapsto (f^*_{\mathbf{A}}(A), \iota_{G, f} \circ f^*_{\mathbf{C}}(c) \circ \iota_{F, f}^{-1}, f^*_{\mathbf{B}}(B)).$$

PROPOSITION 1.1. *The above data determine a presystem of diagram categories*

$$(2) \quad \prod(F, G) : \mathcal{Dia}^{*op} \longrightarrow \mathbf{CAT}.$$

Proof. Let us show that (2) is a functor. For this, consider two composable maps $I \xrightarrow{g} J \xrightarrow{f} K$ in \mathcal{Dia}^* . We have

$$\begin{aligned} \iota_{G, g} f^* \circ g^* (\iota_{G, f} f^* (c) \iota_{F, f}^{-1}) \circ \iota_{F, g}^{-1} f^* = \\ \underbrace{\iota_{G, g} f^* \circ g^* (\iota_{G, f}) \circ g^* (f^* (c))}_{\iota_{G, fg}} \circ \overbrace{g^* (\iota_{F, f}^{-1}) \circ \iota_{F, g}^{-1} f^*}^{\iota_{F, fg}^{-1}} = \iota_{G, fg} (fg)^* (c) \iota_{F, fg}^{-1}. \end{aligned}$$

We see that $(fg)^* = g^* f^* : \prod(F, G)_K \longrightarrow \prod(F, G)_I$.

Since the diagram

$$\begin{array}{ccccccc}
Ff^*(A) & \xrightarrow{\iota_{F,f}^{-1}} & f^*F(A) & \xrightarrow{f^*(c)} & f^*G(B) & \xrightarrow{\iota_{G,f}} & Gf^*(B) \\
F\varphi^* \downarrow & & \varphi^*F \downarrow & & \downarrow \varphi^*G & & \downarrow G\varphi^* \\
Fg^*(A) & \xrightarrow{\iota_{F,g}^{-1}} & g^*F(A) & \xrightarrow{g^*(c)} & g^*G(B) & \xrightarrow{\iota_{G,g}} & Gg^*(B)
\end{array}$$

is commutative for any morphisms $f, g : I \longrightarrow J$ and any bimorphism $\varphi : f \longrightarrow g$ in \mathcal{Dia}^* , the map

$$(f^*(A), \iota_{G,f}f^*(c)\iota_{F,f}^{-1}, f^*(B)) \longmapsto (g^*(A), \iota_{G,g}g^*(c)\iota_{F,g}^{-1}, g^*(B))$$

yields a map φ^* between $f^*, g^* : \prod(F, G)_J \longrightarrow \prod(F, G)_I$. The functoriality axiom is directly verified and left to the reader. \square

Let $\Delta^n = \{0 < \dots < n\} \in \mathcal{Dia}$. If there is no likelihood of confusion, we also denote Δ^0 by 0. Given $I \in \mathcal{Dia}$ and $x \in I$, let $i_{x,I} : 0 \longrightarrow I$ be the functor sending 0 to x . For $A \in \mathbf{C}_I$ let $A_x = i_{x,I}^*A$.

Given $I \in \mathcal{Dia}$ there is a natural functor

$$dia_I : \mathbf{C}_I \longrightarrow \text{Hom}(I, \mathbf{C}_0).$$

It is constructed as follows. For any $x \in I$ we put $dia_I(B)(x) = B_x$. Every morphism $\alpha : x \longrightarrow y$ in I yields a natural transformation $\alpha : i_{x,I} \longrightarrow i_{y,I}$. Then $dia_I(B)(\alpha) := \alpha^* : B_x \longrightarrow B_y$.

Let us consider the following axioms listed below.

Isomorphism Axiom. A morphism $f : A \longrightarrow B$ in \mathbf{C}_I is an isomorphism iff $dia_I(f)$ is so in $\text{Hom}(I, \mathbf{C}_0)$. In other words, it is an isomorphism iff $f_x : A_x \longrightarrow B_x$ is so for all $x \in I$.

Disjoint Union Axiom. (a) If $I = I_1 \coprod I_2$ is a disjoint union of its full subcategories I_1 and I_2 , then the inclusions $i_{1,2} : I_{1,2} \longrightarrow I$ define an equivalence of categories

$$(i_1^*, i_2^*) : \mathbf{C}_I \xrightarrow{\sim} \mathbf{C}_{I_1} \times \mathbf{C}_{I_2}.$$

(b) \mathbf{C}_\emptyset is a trivial category (having precisely one morphism between any pair of objects).

Homotopy Kan Extension Axioms. The left homotopy Kan extension axiom requires that for any functor $f : I \longrightarrow J$, the functor $f^* : \mathbf{C}_J \longrightarrow \mathbf{C}_I$ has a left adjoint $f_! : \mathbf{C}_I \longrightarrow \mathbf{C}_J$. By symmetry, the right homotopy Kan extension axiom says that f^* has a right adjoint $f_* : \mathbf{C}_I \longrightarrow \mathbf{C}_J$. Below we shall also refer to the functors $f_!$ and f_* as left and right homotopy Kan extensions respectively.

In the special case where $f : I^* \longrightarrow 0^*$ comes from the unique functor $I \longrightarrow 0$, we shall write $\underline{\text{Holim}}_I$ for $f_!$ and $\underline{\text{Holim}}_I$ for f_* .

LEMMA 1.2. *Let (f, g) be a pair of adjoint functors in \mathcal{Dia}^* and let*

$$\varphi : fg \longrightarrow 1, \quad \psi : 1 \longrightarrow gf$$

be the adjunction morphisms. Then (f^*, g^*) is a pair of adjoint functors and

$$\varphi^* : g^* f^* \longrightarrow 1, \quad \psi^* : 1 \longrightarrow f^* g^*$$

are the adjunction morphisms.

Proof. We have the following maps

$$f \xrightarrow{f\psi} f g f \xrightarrow{\varphi f} f, \quad g \xrightarrow{\psi g} g f g \xrightarrow{g\varphi} g$$

with $(\varphi f)(f\psi) = 1_f$ and $(g\varphi)(\psi g) = 1_g$. Then,

$$(\varphi f)^*(f\psi)^* = (f^* \varphi^*)(\psi^* f^*) = 1_{f^*}, \quad (g\varphi)^*(\psi g)^* = (\varphi^* g^*)(g^* \psi^*) = 1_{g^*}$$

whence the assertion. \square

DEFINITION. We refer to a functor (1) as a *left (respectively right) system of diagram categories* if the above axioms (the Functoriality Axiom, the Isomorphism Axiom, the Disjoint Union Axiom, and the Left (respectively Right) Homotopy Kan Extension Axiom) are satisfied.

In what follows we shall refer to a left and right system of diagram categories as a *bisystem of diagram categories*.

EXAMPLE. Let \mathcal{C} be a closed model category, and let $I \in \mathcal{D}irf$. There is a natural structure of a closed model category for \mathcal{C}^I (see [6]). Suppose further that \mathcal{C} has a zero object. Denote by $\text{Ho } \mathcal{C}^I$ the homotopy category obtained by inverting the weak equivalences. There is a canonical functor $\mathcal{C}^I \longrightarrow \mathcal{C}^{I^*}$ which extends a I -diagram to I^* by sending the zero object and morphisms in I^* to the zero object and morphisms in \mathcal{C} . Any functor $f : I^* \longrightarrow J^*$ therefore defines a functor $f^* : \mathcal{C}^J \longrightarrow \mathcal{C}^I$. It preserves weak equivalences, hence it defines a functor between homotopy categories. It follows from [6, 1.3.2] that the functor

$$I \in \mathcal{D}irf \longmapsto \text{Ho } \mathcal{C}^I$$

determines a bisystem of diagram categories of the domain $\mathcal{D}irf$.

Given an arbitrary model category \mathcal{C} , let \mathcal{C}_* denote the model category under the terminal object $*$ (see [10, p. 4]). Then \mathcal{C}_* is pointed. Its bisystem of diagram categories of the domain $\mathcal{D}irf$ is, by definition, that associated to \mathcal{C}_* .

Let $F : \mathbf{A} \longrightarrow \mathbf{C}$ be a morphism between two left systems of diagram categories \mathbf{A} and \mathbf{C} , and let $f : I \longrightarrow J$ be a map in $\mathcal{D}ia^*$. Consider the adjunction maps

$$\alpha : 1 \longrightarrow f^* f_! \quad \text{and} \quad \beta : f_! f^* \longrightarrow 1.$$

Denote by $\gamma_{F,f}$ the composed map

$$f_! F \xrightarrow{f_! F \alpha} f_! F f^* f_! \xrightarrow{f_! \iota_{F,f}^{-1} f_!} f_! f^* F f_! \xrightarrow{\beta F f_!} F f_!.$$

We say that F is *right exact* if $\gamma_{F,f}$ is an isomorphism and if the following two compatibility relations hold:

$$(3) \quad F \alpha_{\mathbf{A}} = \iota_{F,f} f_! \circ f^*(\gamma_{F,f}) \circ \alpha_{\mathbf{C}} F \quad \text{and} \quad F \beta_{\mathbf{A}} = \beta_{\mathbf{C}} F \circ f_!(\iota_{F,f}^{-1}) \circ \gamma_{F,f}^{-1} f^*.$$

That is, $F\alpha_{\mathbf{A}}$ is the composite

$$F \xrightarrow{\alpha_{\mathbf{C}}F} f^* f_! F \xrightarrow{f^*(\gamma_{F,f})} f^* F f_! \xrightarrow{\iota_{F,f} f_!} F f^* f_!$$

and $F\beta_{\mathbf{A}}$ is the composite

$$F f_! f^* \xrightarrow{\gamma_{F,f}^{-1} f^*} f_! F f^* \xrightarrow{f_!(\iota_{F,f}^{-1})} f_! f^* F \xrightarrow{\beta_{\mathbf{C}}F} F.$$

The notion of a left exact (respectively exact) morphism between two right systems of diagram categories (respectively between two bisystems of diagram categories) is similarly defined.

PROPOSITION 1.3. *Let $F : \mathbf{A} \longrightarrow \mathbf{C}$ and $G : \mathbf{B} \longrightarrow \mathbf{C}$ be two right exact (respectively left exact) morphisms between left system of diagram categories (respectively right system of diagram categories); then the fibred product $\prod(F, G)$ is a left system of diagram categories (respectively right system of diagram categories) as well.*

Proof. We prove the assertion for left systems of diagram categories. The case of a right system of diagram categories is proved by symmetry. By Proposition 1.1 $\prod(F, G)$ is a presystem of diagram categories. Obviously, it satisfies both the isomorphism axiom and the disjoint union axiom. We must thus verify the homotopy Kan extension axioms.

Let $f : I \longrightarrow J$ be a map in $\mathcal{D}ia^*$. We define the functor

$$f_! : \prod(F, G)_I \longrightarrow \prod(F, G)_J$$

as follows:

$$(A, c, B) \longmapsto (f_!(A), \gamma_{G,f} f_!(c) \gamma_{F,f}^{-1}, f_!(B)).$$

Then the adjunction maps $\alpha_{\mathbf{A}, \mathbf{B}} : 1 \longrightarrow f^* f_!$ and $\beta_{\mathbf{A}, \mathbf{B}} : f_! f^* \longrightarrow 1$ determine those for $\prod(F, G)$. To see this we have to check that the squares

$$\begin{array}{ccc} FA & \xrightarrow{c} & GB \\ F\alpha_{\mathbf{A}} \downarrow & & \downarrow G\alpha_{\mathbf{B}} \\ Ff^* f_! A & \xrightarrow{c'} & Gf^* f_! B \end{array}$$

with $c' = \iota_{G,f} f_! \circ f^*(\gamma_{G,f}) \circ f^* f_!(c) \circ f^*(\gamma_{F,f}^{-1}) \circ \iota_{F,f}^{-1} f_!$ and

$$\begin{array}{ccc} Ff_! f^* A & \xrightarrow{c''} & Gf_! f^* B \\ F\beta_{\mathbf{A}} \downarrow & & \downarrow G\beta_{\mathbf{B}} \\ FA & \xrightarrow{c} & GB \end{array}$$

with $c'' = \gamma_{G,f} f^* \circ f_!(\iota_{G,f}) \circ f_! f^*(c) \circ f_!(\iota_{F,f}^{-1}) \circ \gamma_{F,f}^{-1} f^*$ are commutative.

We have the following commutative diagram.

$$\begin{array}{ccc}
FA & \xrightarrow{c} & GB \\
\alpha_{\mathbf{C}} F \downarrow & & \downarrow \alpha_{\mathbf{C}} G \\
f^* f_! FA & \xrightarrow{f^* f_!(c)} & f^* f_! GB \\
\downarrow \iota_{F,f} f_! \circ f^*(\gamma_{F,f}) & & \downarrow \iota_{G,f} f_! \circ f^*(\gamma_{G,f}) \\
F f^* f_! A & \xrightarrow{c'} & G f^* f_! B
\end{array}$$

In view of the relation (3) one obtains

$$\begin{aligned}
G\alpha_{\mathbf{B}} \circ c &= \iota_{G,f} f_! \circ f^*(\gamma_{G,f}) \circ \alpha_{\mathbf{C}} G \circ c = \iota_{G,f} f_! \circ f^*(\gamma_{G,f}) \circ f^* f_!(c) \circ \alpha_{\mathbf{C}} F = \\
&= \iota_{G,f} f_! \circ f^*(\gamma_{G,f}) \circ f^* f_!(c) \circ f^*(\gamma_{F,f}^{-1}) \circ \iota_{F,f}^{-1} f_! \circ F\alpha_{\mathbf{A}} = c' \circ F\alpha_{\mathbf{A}}.
\end{aligned}$$

So, the first square is commutative. Commutativity of the second square is similarly proved. It is routine to check that both the following composites are the identities (of $f_{\prod(F,G)}^*$, respectively $f_{\prod(F,G)}!$).

$$f^* \xrightarrow{\alpha f^*} f^* f_! f^* \xrightarrow{f^* \beta} f^*, \quad f_! \xrightarrow{f_! \alpha} f_! f^* f_! \xrightarrow{\beta f_!} f_!.$$

This yields the homotopy Kan extension axiom. \square

1.3. Consequences of the axioms. In this section we discuss some consequences of the axioms. We also refer the reader to Franke's work [6].

1.3.1. Properties of the homotopy Kan extension functors. A map $f : I \longrightarrow J$ in *Dia* is a *closed (open) immersion* if it is a fully faithful inclusion such that for any $x \in J$ the relation $\text{Hom}(I, x) \neq \emptyset$ ($\text{Hom}(x, I) \neq \emptyset$) implies $x \in I$. The following is straightforward.

THE IMMERSION LEMMA. *Let $f : I \longrightarrow J$ be a closed (respectively open) immersion in *Dia*. Then the map $g : J^* \longrightarrow I^*$ taking $j \in J$ to j if $j \in I$ and to \star otherwise is a right (respectively left) adjoint to f^* .*

PROPOSITION 1.4. *Suppose \mathbf{C} is a left system of diagram categories. Let $f : I \longrightarrow J$ be a functor, $x \in J$, and let*

$$\begin{aligned}
i_x &: J/x \longrightarrow J \\
j_x &: f/x \longrightarrow I \\
l &: f/x \longrightarrow J/x
\end{aligned}$$

be the canonical functors. If J is a poset then for $A \in \mathbf{C}_I$ we have isomorphisms

$$\begin{aligned}
(f_! A)_x &\simeq \underline{\text{Holim}}_{J/x} i_x^* f_! A \\
&\simeq \underline{\text{Holim}}_{J/x} l_! j_x^* A \\
&\simeq \underline{\text{Holim}}_{f/x} j_x^* A.
\end{aligned}$$

If \mathbf{C} is a right system of diagram categories, a dual assertion holds for projective homotopy limits and right homotopy Kan extensions.

Proof. Let $c : 0 \longrightarrow J$ and $d : 0 \longrightarrow J/x$ be the functors taking 0 to x and (x, id) respectively. It follows that d is a right adjoint to $J/x \longrightarrow 0$, and hence d^* is isomorphic to $\underline{\text{Holim}}_{J/x}$. We then have

$$f_! A_x = c^* f_! A = d^* i_x^* f_! A \simeq \underline{\text{Holim}}_{J/x} i_x^* f_! A$$

whence the first isomorphism follows.

Since J is a poset, i_x is an open immersion and so i_x^* has a left adjoint i_{x+} by the immersion lemma. It sends every object $y \in J$ to $(y, y \leq x)$ if $y \leq x$, and to $(\star, \star \leq x)$ otherwise. Similarly, j_x^* has a left adjoint j_{x+} which sends $y \in I$ to $(\star, \star \leq x)$ if $f(y) \not\leq x$, and to $(y, f(y) \leq x)$ otherwise. It follows that $i_{x+!}$ and $j_{x+!}$ are isomorphic to i_x^* and j_x^* respectively. Since $l j_{x+} = i_{x+} f$, we see that $l_! j_{x+!} \simeq i_{x+!} f_!$, and hence $l_! j_x^*$ is isomorphic to $i_x^* f_!$. This implies the second isomorphism. The last isomorphism is obvious. \square

In the case where f is the inclusion of a full subcategory $I \subseteq J$ with J a poset, then for every object $x \in I$ the category f/x has a final object (x, id_x) . Therefore we have the following

COROLLARY 1.5. *Let \mathbf{C} be a left system of diagram categories (respectively right system of diagram categories), and let $f : I \longrightarrow J$ be the inclusion of a full subcategory with J a poset. Then the canonical morphism in \mathbf{C}_I*

$$A \longrightarrow f^* f_! A \quad (\text{respectively } f^* f_* A \longrightarrow A)$$

is an isomorphism for every object A of \mathbf{C}_I .

The last proposition can often be used to reduce assertions about the functors $f_!$ and f_* to the similar assertions about $\underline{\text{Holim}}_J$ and $\underline{\text{Holim}}_J$. The following proposition is concerned with the question of replacing J by a smaller category.

PROPOSITION 1.6. *Suppose \mathbf{C} is a left system of diagram categories. Let $i : I^* \longrightarrow J^*$ be some functor (typically the inclusion of a subcategory). If i has a left adjoint of the form l^* with $l : J \longrightarrow I$, then $\underline{\text{Holim}}_J A \simeq \underline{\text{Holim}}_I i^* A$. Dually, suppose \mathbf{C} is a right system of diagram categories. If i has a right adjoint of the form r^* for some functor $r : J \longrightarrow I$, then $\underline{\text{Holim}}_J A \simeq \underline{\text{Holim}}_I i^* A$.*

Proof. (1) Let l be a left adjoint of i . Then $l_!$ is naturally isomorphic to i^* . It follows that $\underline{\text{Holim}}_J = \underline{\text{Holim}}_I \circ l_! \simeq \underline{\text{Holim}}_I \circ i^*$. \square

1.3.2. Cartesian squares. Let $\square \in \mathcal{D}ia$ be the poset $\Delta^1 \times \Delta^1$, possessing the following elements:

$$\begin{array}{ccc} (0, 0) & \longrightarrow & (0, 1) \\ \downarrow & & \downarrow \\ (1, 0) & \longrightarrow & (1, 1) \end{array}$$

where \longrightarrow stands for $<$. Let $\ulcorner \subset \square$ be the subposet obtained by removing the lower right corner $(1, 1)$, and let $\lrcorner \subset \square$ be the subposet containing all elements of \square save for $(0, 0)$. Let $i_\ulcorner : \ulcorner \longrightarrow \square$ and $i_\lrcorner : \lrcorner \longrightarrow \square$ be the inclusions. Let \mathbf{C} be a

right system of diagram categories (respectively left system of diagram categories). An object A of \mathbf{C}_\square is called *cartesian* (respectively *cocartesian*) iff the canonical morphism $A \longrightarrow i_{\lrcorner*} i_{\lrcorner}^* A$ is an isomorphism (respectively iff the canonical morphism $i_{\lrcorner!} i_{\lrcorner}^* A \longrightarrow A$ is an isomorphism).

LEMMA 1.7. *Let \mathbf{C} be a left system of diagram categories. An object A of \mathbf{C}_\square is cocartesian if and only if $A_{(1,1)} \simeq \underline{\text{Holim}}_{\rightarrow} i_{\lrcorner}^* A$. Dually, if \mathbf{C} is a right system of diagram categories, then an object A of \mathbf{C}_\square is cartesian if and only if $A_{(0,0)} \simeq \underline{\text{Holim}}_{\leftarrow} i_{\lrcorner}^* A$.*

Proof. Let A be an arbitrary object of \mathbf{C}_\square . By Corollary 1.5 it follows that a natural morphism

$$i_{\lrcorner}^* A \longrightarrow i_{\lrcorner}^* i_{\lrcorner!} i_{\lrcorner}^* A$$

is an isomorphism. Therefore $A_{(i,j)} \simeq i_{\lrcorner!} i_{\lrcorner}^* A_{(i,j)}$ if $(i,j) \in \{(0,0), (0,1), (1,0)\}$.

It follows from Proposition 1.4 that

$$i_{\lrcorner!} i_{\lrcorner}^* A_{(1,1)} \simeq \underline{\text{Holim}}_{\rightarrow} i_{\lrcorner}^* A.$$

Now our assertion immediately follows. \square

From now on all left or right systems of diagram categories are assumed to be of the domain Ord^* . Let an object A of $\mathbf{C}_{I \times \square}$, $I \in \text{Ord}$, be called cartesian if

$$A \longrightarrow (\text{id}_I \times i_{\lrcorner})_*(\text{id}_I \times i_{\lrcorner})^* A$$

is an isomorphism, and let cocartesiannes be similarly defined, replacing a right system of diagram categories \mathbf{C} by a left system of diagram categories, \lrcorner by \lrcorner and $(\text{id}_I \times i_{\lrcorner})_*$ by $(\text{id}_I \times i_{\lrcorner})_!$ and reversing the direction of the arrow. It follows from the isomorphism axiom and from Proposition 1.8 below that an object $A \in \mathbf{C}_{I \times \square}$ is cartesian (respectively cocartesian) iff the object $A_{x,\square} = (i_{x,I} \times \text{id}_{\square})^* A$ is cartesian (respectively cocartesian) in \mathbf{C}_\square for all $x \in I$.

For any object I of Ord , we denote by $\mathbf{C}(I)$ the left system of diagram categories defined as $\mathbf{C}(I)_J = \mathbf{C}_{I \times J}$. Here I plays the role of a parameter.

PROPOSITION 1.8. *Let \mathbf{C} be a left system of diagram categories. Let $f : I \longrightarrow J$ be a map in Ord^* . Then f gives the right exact functor*

$$f^* : \mathbf{C}(J) \longrightarrow \mathbf{C}(I)$$

induced by $(f \times 1_K)^ : \mathbf{C}_{J \times K} \longrightarrow \mathbf{C}_{I \times K}$ with $K \in \text{Ord}^*$. In particular, the functor respects cocartesian squares. A dual assertion also holds for right systems of diagram categories.*

Proof. Given a map $g : K \longrightarrow L$ in Ord^* we have to show that a natural morphism

$$\gamma : (1_I \times g)_!(f \times 1_K)^* \longrightarrow (f \times 1_L)^*(1_J \times g)_!$$

is an isomorphism. Let $A \in \mathbf{C}_{J \times K}$ and $(x,y) \in I \times L$.

The map $\varphi_{x,y} : 1 \times g/(x,y) \longrightarrow g/y, ((u,u \longrightarrow x), (v,g(v) \longrightarrow y)) \longmapsto (v,g(v) \longrightarrow y)$, has a right adjoint $\psi_{x,y} : g/y \longrightarrow 1 \times g/(x,y), (v,g(v) \longrightarrow y) \longmapsto ((x,x = x), (v,g(v) \longrightarrow y))$. It follows from Proposition 1.6 that $\underline{\text{Holim}}_{\rightarrow 1 \times g/(x,y)} \simeq \underline{\text{Holim}}_{\rightarrow g/y} \psi_{x,y}^*$.

By Proposition 1.4 it follows that

$$\begin{aligned} (1_I \times g)!(f \times 1_K)^* A_{(x,y)} &\simeq \underline{\text{Holim}}_{1_I \times g/(x,y)} j_{(x,y)}^* (f \times 1_K)^* A \\ &\simeq \underline{\text{Holim}}_{g/y} \psi_{x,y}^* j_{(x,y)}^* (f \times 1_K)^* A. \end{aligned}$$

On the other hand,

$$\begin{aligned} (f \times 1_L)^* (1_J \times g) A_{(x,y)} &= i_{(f(x),y)}^* (1_J \times g) A \simeq \underline{\text{Holim}}_{1_J \times g/(f(x),y)} j_{(f(x),y)}^* A \\ &\simeq \underline{\text{Holim}}_{g/y} \psi_{f(x),y}^* j_{(f(x),y)}^* A = \underline{\text{Holim}}_{g/y} \psi_{x,y}^* j_{(x,y)}^* (f \times 1_K)^* A. \end{aligned}$$

We have used here the relation $j_{(f(x),y)} \psi_{f(x),y} = (f \times 1_K) j_{(x,y)} \psi_{x,y}$. So γ is an isomorphism. It is routine to check that the functor of the proposition satisfies the compatibility relations (3). \square

DEFINITION. Let \mathbf{C} be a right system of diagram categories (respectively left system of diagram categories). A *square* in I is a functor $i : \square \rightarrow I$ which is injective on the set of objects. Let A be an object of \mathbf{C}_I ; we say that A *makes the square cartesian* (respectively *cocartesian*) if $i^* A$ is cartesian (respectively cocartesian).

PROPOSITION 1.9. *Let \mathbf{C} be a left system of diagram categories, and let i be a square in a poset I . If the functor $\Gamma \rightarrow (I - i(1,1)/i(1,1))$ possesses a left adjoint and if $A = f_! B$ with $f : J \rightarrow I$ a functor not containing $i(1,1)$ in its image, then A makes i cocartesian. Let \mathbf{C} be a right system of diagram categories. The same holds if the functor $\sqcup \rightarrow (I - i(0,0) \setminus i(0,0))$ possesses a right adjoint and if $A = f_* B$ with $f : J \rightarrow I$ a functor not containing $i(0,0)$ in its image.*

Proof. It suffices to prove the first assertion. Let $j : I - i(1,1) \rightarrow I$ be the inclusion. By our assumption on the image of f , it factors as $J \xrightarrow{g} I - i(1,1) \xrightarrow{j} I$. From Corollary 1.5 it follows that $j^* A = j^* j_! g_! B \simeq g_! B$, whence $A = j_! g_! B \simeq j_! j^* A$. We have

$$i^* A_{(1,1)} = A_{i(1,1)} \simeq \underline{\text{Holim}}_{I - i(1,1)/i(1,1)} h^* A \simeq \underline{\text{Holim}}_{i_r} i_r^* i^* A,$$

by Proposition 1.4 and Proposition 1.6(1), where

$$h : I - i(1,1)/i(1,1) \rightarrow I$$

is the canonical functor. Our assertion follows now from Lemma 1.7. \square

PROPOSITION 1.10. *(Concatenation of squares and (co-)cartesiannes) Let \mathbf{C} be a left system of diagram categories (respectively right system of diagram categories), and let $d_{0,1,2} : \Delta^1 \rightarrow \Delta^2$ be the three monotonic injections, and $A \in \mathbf{C}_{\Delta^1 \times \Delta^2}$. Suppose that $(d_2 \times \text{id}_{\Delta^1})^* A \in \mathbf{C}_{\square}$ is cocartesian (respectively $(d_0 \times \text{id}_{\Delta^1})^* A \in \mathbf{C}_{\square}$ is cartesian). Then $(d_0 \times \text{id}_{\Delta^1})^* A$ is cocartesian (respectively $(d_2 \times \text{id}_{\Delta^1})^* A \in \mathbf{C}_{\square}$ is cartesian) if and only if $(d_1 \times \text{id}_{\Delta^1})^* A$ is cocartesian (respectively $(d_1 \times \text{id}_{\Delta^1})^* A \in \mathbf{C}_{\square}$ is cartesian).*

Proof. Let $I = \Delta^1 \times \Delta^2 - (1,2)$ and $J = I - (1,1)$, and let j and k be the inclusions of the subposets I and J into $\Delta^1 \times \Delta^2$, and let $l : J \rightarrow I$ be the inclusion. It

follows from Proposition 1.9 that $k_!k^*A$ makes both visible squares in $\Delta^1 \times \Delta^2$

$$\begin{array}{ccccc} (0, 0) & \longrightarrow & (0, 1) & \longrightarrow & (0, 2) \\ \downarrow & & \downarrow & & \downarrow \\ (1, 0) & \longrightarrow & (1, 1) & \longrightarrow & (1, 2) \end{array}$$

cocartesian. Proposition 1.4 then implies the following isomorphism,

$$k_!k^*A_{(1,1)} \simeq \underline{\mathrm{Holim}}_r i_r^*(d_2 \times \mathrm{id}_{\Delta^1})^*A.$$

By Proposition 1.9 we also have

$$k_!k^*A_{(1,2)} \simeq \underline{\mathrm{Holim}}_r i_r^*(d_1 \times \mathrm{id}_{\Delta^1})^*A.$$

By our assumption on $(d_2 \times \mathrm{id}_{\Delta^1})^*A$, it follows that $j^*A \simeq l_!k^*A$ and that $(d_1 \times \mathrm{id}_{\Delta^1})^*A$ is cocartesian iff A is isomorphic to $k_!k^*A = j_!l_!k^*A \simeq j_!j^*A$. By Proposition 1.9 $j_!j^*A$ makes the right square cocartesian. Since $j_!j^*A_{(1,2)} \simeq \underline{\mathrm{Holim}}_r i_r^*(d_0 \times \mathrm{id}_{\Delta^1})^*A$ this is the case iff $(d_0 \times \mathrm{id}_{\Delta^1})^*A$ is cocartesian. \square

PROPOSITION 1.11. *Let \mathbf{C} be a left system of diagram categories (respectively right system of diagram categories). For every I , the category \mathbf{C}_I has a zero-object and finite coproducts (respectively products). For every functor $f : I \longrightarrow J$ the functor $f_!$ (respectively f_*) preserves coproducts (respectively products).*

Proof. Let $f : I^* \longrightarrow \emptyset^*$ be the unique functor. The inclusion $g : \emptyset^* \longrightarrow I^*$ is left and right adjoint to f . It follows that g^* is left and right adjoint to f^* . Therefore given $0 \in \mathbf{C}_\emptyset$ the object f^*0 (denote it also by 0) is a zero-object in \mathbf{C}_I .

Let $I \amalg I$ be the disjoint union of two copies of I , and let $p : I \amalg I \longrightarrow I$ be the functor which is the identity on both copies of I . By the disjoint union axiom $\mathbf{C}_{I \amalg I} \simeq \mathbf{C}_I \times \mathbf{C}_I$. Hence, the functor $p_!$ provides the coproduct. Since $f_!$ is left adjoint to f^* with $f : I \longrightarrow J$ a map in $\mathcal{D}ia$, it preserves coproducts. \square

Let $f : I^* \longrightarrow J^*$ be a map in $\mathcal{D}ia^*$ and $x \in I$. If $f(x) = \star$ then $f^*A_x = i_{x,I}^*f^*A = 0$ for any $A \in \mathbf{C}_J$. Indeed, the composite $0^* \xrightarrow{i_{x,I}} I^* \xrightarrow{f} J^*$ factors as $0^* \xrightarrow{j} \emptyset^* \xrightarrow{l} J^*$, whence $f^*A_x = j^*l^*A = 0$.

2. DÉRIVATEURS

2.1. Definitions. Let $\mathcal{D}ia$ be a category of diagrams. So far we considered only functors

$$\mathbf{C} : \mathcal{D}ia^{\mathrm{op}} \longrightarrow \mathbf{CAT}$$

evaluated on the category $\mathcal{D}ia^*$. The horizontal morphisms $I \longrightarrow J$ in $\mathcal{D}ia^*$ are given by the functors $I^* \longrightarrow J^*$ mapping \star to \star . It is also of particular interest to consider functors

$$(4) \quad \mathbf{D} : \mathcal{D}ia^{\mathrm{op}} \longrightarrow \mathbf{CAT}$$

evaluated on the underlying category $\mathcal{D}ia$. Here we follow the terminology of [14].

All the axioms of section 1 can also be reformulated for morphisms and bimorphisms in $\mathcal{D}ia$.

DEFINITION. A functor (4) is called a *predérivateur* if it satisfies the Functoriality Axiom. It is a *left (respectively right) dérivateur* if the Functoriality Axiom, the Isomorphism Axiom, the Disjoint Union Axiom, the Left (respectively Right) Homotopy Kan Extension Axiom, and the Left (respectively Right) Base Change Axiom below are satisfied.

Base Change Axiom. Let $f : I \longrightarrow J$ be a morphism in $\mathcal{D}ia$ and $x \in J$. Consider the diagram in $\mathcal{D}ia$

$$\begin{array}{ccc} f/x & \xrightarrow{j_x} & I \\ p \downarrow & \swarrow \alpha_x & \downarrow f \\ 0 & \xrightarrow{i_{x,J}} & J \end{array}$$

with j_x a natural map and α_x the bimorphism

$$f j_x \longrightarrow i_{x,I} p, \quad \alpha_x : f j_x(y, a : f(y) \longrightarrow x) = f(y) \xrightarrow{a} x = i_{x,J} p(y, a).$$

The bimorphism α_x induces a bimorphism $\beta_x : p_! j_x^* \longrightarrow i_{x,I}^* f_!$ which is the composite

$$p_! j_x^* \longrightarrow p_! j_x^* f^* f_! \xrightarrow{p_! \alpha_x f_!} p_! p^* i_x^* f_! \longrightarrow i_x^* f_!.$$

The left base change axiom requires β_x to be an isomorphism.

By symmetry, the right base change axiom says that the diagram

$$\begin{array}{ccc} f \setminus x & \xrightarrow{l_x} & I \\ q \downarrow & \swarrow \gamma_x & \downarrow f \\ 0 & \xrightarrow{i_{x,J}} & J \end{array}$$

yields an isomorphism $\delta_x : i_{x,I}^* f_* \longrightarrow q_* l_x^*$.

We shall refer to a left and right dérivateur as a *bidérivateur*.

EXAMPLE. Given a category \mathcal{C} closed under colimits, then the representable predérivateur associated to \mathcal{C} is a left dérivateur. A typical example of a bidérivateur (of the domain $\mathcal{D}irf$) is given by the functor

$$I \longmapsto \mathrm{Ho} \mathcal{C}^I$$

with \mathcal{C} a closed model category (see [3] for details).

From now on all left or right dérivateurs are assumed to be of the domain $\mathcal{D}ia$. The notions of a morphism between two predérivateurs, of the fibred product of a pair of morphisms are defined similar to those for presystems of diagram categories. It is similarly proved that the fibred product of two morphisms is a predérivateur and that it is a left (right) dérivateur whenever both morphisms are right (left) exact.

PROPOSITION 2.1. *Suppose \mathbf{D} is a left dérivateur. Let $f : I \longrightarrow J$ be a functor in $\mathcal{D}ia$, $x \in J$. Then for $A \in \mathbf{D}_I$ we have an isomorphism*

$$(f_! A)_x \simeq \underline{\mathrm{Holim}}_{f/x} j_x^* A.$$

If \mathbf{D} is a right *dérivateur*, a dual assertion holds for projective homotopy limits and right homotopy Kan extensions.

Proof. The assertion immediately follows from the base change axiom. Precisely,

$$\beta_x^{-1} : (f_! A)_x = i_{x,J}^* f_! A \longrightarrow \underline{\mathrm{Holim}}_{f/x} j_x^* A$$

is an isomorphism (recall the reader that $\underline{\mathrm{Holim}}_{f/x} = p_!$ by definition). \square

In the case where f is the inclusion of a full subcategory $I \subseteq J$, then for every object $x \in I$ the category f/x has a final object (x, id_x) . Therefore we have the following

COROLLARY 2.2. *Let \mathbf{D} be a left *dérivateur* (respectively right *dérivateur*), and let $f : I \longrightarrow J$ be the inclusion of a full subcategory. Then the canonical morphism in \mathbf{D}_I*

$$A \longrightarrow f^* f_! A \quad (\text{respectively } f^* f_* A \longrightarrow A)$$

is an isomorphism for every object A of \mathbf{D}_I .

The notion of a (co-)cartesian square is defined as above. Below we formulate similar statements about (co-)cartesian squares without proofs. They repeat those in the preceding section word for word.

LEMMA 2.3. *Let \mathbf{D} be a left *dérivateur*. An object A of \mathbf{D}_\square is cocartesian if and only if $A_{(1,1)} \simeq \underline{\mathrm{Holim}}_{\rightarrow} i_r^* A$. Dually, if \mathbf{D} is a right *dérivateur*, then an object A of \mathbf{D}_\square is cartesian if and only if $A_{(0,0)} \simeq \underline{\mathrm{Holim}}_{\leftarrow} i_l^* A$.*

For any object I of \mathbf{Dia} , we denote by $\mathbf{D}(I)$ the left *dérivateur* defined as $\mathbf{D}(I)_J = \mathbf{D}_{I \times J}$.

PROPOSITION 2.4. *Let \mathbf{D} be a left *dérivateur*. Let $f : I \longrightarrow J$ be a map in \mathbf{Dia} . Then f gives the right exact functor*

$$f^* : \mathbf{D}(J) \longrightarrow \mathbf{D}(I)$$

induced by $(f \times 1_K)^ : \mathbf{D}_{J \times K} \longrightarrow \mathbf{D}_{I \times K}$ with $K \in \mathbf{Dia}$. In particular, the functor respects cocartesian squares. A dual assertion also holds for right *dérivateurs*.*

PROPOSITION 2.5. *Let \mathbf{D} be a left *dérivateur*, and let i be a square in a category $I \in \mathbf{Dia}$. If the functor $\Gamma \longrightarrow (I - i(1,1)/i(1,1))$ possesses a left adjoint and if $A = f_! B$ with $f : J \longrightarrow I$ a functor not containing $i(1,1)$ in its image, then A makes i cocartesian. Let \mathbf{D} be a right *dérivateur*. The same holds if the functor $\lrcorner \longrightarrow (I - i(0,0) \setminus i(0,0))$ possesses a right adjoint and if $A = f_* B$ with $f : J \longrightarrow I$ a functor not containing $i(0,0)$ in its image.*

PROPOSITION 2.6. *(Concatenation of squares and (co-)cartesiennes) Let \mathbf{D} be a left *dérivateur*. (respectively right *dérivateur*), and let $d_{0,1,2} : \Delta^1 \longrightarrow \Delta^2$ be the three monotonic injections, and $A \in \mathbf{D}_{\Delta^1 \times \Delta^2}$. Suppose that $(d_2 \times \mathrm{id}_{\Delta^1})^* A \in \mathbf{D}_\square$ is cocartesian (respectively $(d_0 \times \mathrm{id}_{\Delta^1})^* A \in \mathbf{D}_\square$ is cartesian). Then $(d_0 \times \mathrm{id}_{\Delta^1})^* A$ is cocartesian (respectively $(d_2 \times \mathrm{id}_{\Delta^1})^* A \in \mathbf{D}_\square$ is cartesian) if and only if $(d_1 \times \mathrm{id}_{\Delta^1})^* A$ is cocartesian (respectively $(d_1 \times \mathrm{id}_{\Delta^1})^* A \in \mathbf{D}_\square$ is cartesian).*

PROPOSITION 2.7. *Let \mathbf{D} be a left *dérivateur* (respectively right *dérivateur*). For every I , the category \mathbf{D}_I has an initial (respectively final) object and finite coproducts (respectively products). For every functor $f : I \longrightarrow J$ the functor $f_!$ (respectively f_*) preserves coproducts (respectively products).*

Proof. Let $f : \emptyset \longrightarrow I$ be the inclusion and $0 \in \mathbf{D}_\emptyset$. Since $f_!$ is a left adjoint to f^* , it follows that $f_!0$ (denote it also by 0) is an initial object in \mathbf{D}_I .

Let $I \amalg I$ be the disjoint union of two copies of I , and let $p : I \amalg I \longrightarrow I$ be the functor which is the identity on both copies of I . By the disjoint union axiom $\mathbf{D}_{I \amalg I} \simeq \mathbf{D}_I \times \mathbf{D}_I$. Hence, the functor $p_!$ provides the coproduct. Since $f_!$ is left adjoint to f^* with $f : I \longrightarrow J$ a map in $\mathcal{D}ia$, it preserves coproducts and f^* preserves products (whenever they exist). \square

2.2. Pointed *dérivateurs*. The *dérivateurs* we work with below must also satisfy some extra conditions. We start with definitions.

DEFINITION. A left *dérivateur* is said to be *pointed* if it satisfies the following three conditions:

- (1) for any closed immersion $f : I \longrightarrow J$ in $\mathcal{D}ia$, the structure functor $f_!$ possesses a left adjoint $f^?$;
- (2) for any open immersion $f : I \longrightarrow J$ in $\mathcal{D}ia$, the structure functor f^* possesses a right adjoint f_* ;
- (3) for any open immersion $f : I \longrightarrow J$ in $\mathcal{D}ia$ and any object $x \in J$, the base change morphism of the diagram

$$\begin{array}{ccc} f \setminus x & \xrightarrow{l_x} & I \\ q \downarrow & \nearrow \gamma_x & \downarrow f \\ 0 & \xrightarrow{i_{x,J}} & J \end{array}$$

yields an isomorphism $\delta_x : i_{x,I}^* f_* \longrightarrow q_* l_x^*$.

The corresponding notion for a right *dérivateur* to be pointed is defined by symmetry.

We note that for any open immersion $f : I \longrightarrow J$ in $\mathcal{D}ia$ and any object $x \in J$ a right adjoint q_* with $q : f \setminus x \longrightarrow 0$ the unique map always exists. Indeed, if x is not in I then $f \setminus x = \emptyset$ and q_* exists, because $\emptyset \longrightarrow 0$ is an open immersion. If $x \in I$ then $f \setminus x$ has an initial object $(x, x = x)$ and we put $q_* = p^*$ with $0 \xrightarrow{p} (x, x = x) \in f \setminus x$.

Let \mathbf{D} be a left (right) pointed *dérivateur*. Then \mathbf{D}_I has a zero object for any $I \in \mathcal{D}ia$. For the inclusion $\emptyset \longrightarrow I$ is both a closed and an open immersion, and therefore $0 = f_!0$ ($0 = f_*0$) is also a final (initial) object. Also, it follows that for every open immersion $f : I \longrightarrow J$ in $\mathcal{D}ia$ and any object $x \in J$ the “value” $f_* A_x$ at x , $A \in \mathbf{D}_I$, equals either to A_x if $x \in I$ or to 0 otherwise.

In what follows, we refer to a left and right pointed *dérivateur* as a *pointed bidérivateur*.

2.3. Examples. Given $I \in \mathcal{D}irf$ and a Waldhausen category \mathcal{C} , the functor category \mathcal{C}^I is a Waldhausen category, too. A map $F \longrightarrow G$ in \mathcal{C}^I is a cofibration (respectively weak equivalence) if $F(x) \longrightarrow G(x)$ is so for every $x \in I$.

DEFINITION. I. Let \mathcal{A} be a category with finite coproducts and an initial object e . Assume that \mathcal{A} has two distinguished classes of maps, called *weak equivalences* and *cofibrations*. A map is called a *trivial cofibration* if it is both a weak equivalence and a cofibration. We call \mathcal{A} a *category of cofibrant objects* if the following axioms are satisfied.

(A) Let f and g be maps such that gf is defined. If two of f , g , gf are weak equivalences then so the third. Any isomorphism is a weak equivalence.

(B) The composite of two cofibrations is a cofibration. Any isomorphism is a cofibration.

(C) Given a diagram

$$A \xleftarrow{u} C \xrightarrow{v} B,$$

with v a cofibration (respectively a trivial cofibration), the pushout $A \coprod_C B$ exists and the map $A \rightarrow A \coprod_C B$ is a cofibration (respectively a trivial cofibration).

(D) Any map u in \mathcal{A} can be factored $u = pi$ with p a weak equivalence and i a cofibration.

(E) For any object A the map $e \rightarrow A$ is a cofibration.

Note that \mathcal{A} is a category of cofibrant objects in the sense of Brown [2].

For instance, the Waldhausen category of bounded complexes $C^b(\mathcal{E})$ of an exact category \mathcal{E} with weak equivalences quasi-isomorphisms and cofibrations componentwise admissible monomorphisms is a category of cofibrant objects.

Let \mathcal{C} be a Waldhausen category of cofibrant objects and let $\text{Ho } \mathcal{C}$ denote the category obtained from \mathcal{C} by inverting weak equivalences. One can define the notion of the homotopy for two maps f and g (see [2]) and then the category $\pi\mathcal{C}$ with the same objects as \mathcal{C} and with $\pi\mathcal{C}(A, B)$ by equal to the quotient of $\mathcal{C}(A, B)$ by the equivalence relation $f \sim g$ defined in terms of the homotopy. Then the class of weak equivalences in $\pi\mathcal{C}$ admits a calculus of left fractions [2]. Given $I \in \text{Dirf}$, it follows from [4, 1.31] that the functor category \mathcal{C}^I is a Waldhausen category of cofibrant objects.

THEOREM 2.8 (Cisinski [4]). *If \mathcal{C} is a Waldhausen category of cofibrant objects, then the hyperfunctor*

$$\mathbf{D}\mathcal{C} : I \in \text{Dirf} \mapsto \mathbf{D}\mathcal{C}_I = \text{Ho } \mathcal{C}^I$$

determines a left pointed dérivateur of the domain Dirf .

3. THE S .-CONSTRUCTION

Throughout this section \mathbf{B} is assumed to be either a left system of diagram categories (of the domain Ord^*) or a left pointed dérivateur (of the domain Dia). Let $\text{Ar } \Delta^n$ be the poset of pairs (i, j) , $0 \leq i \leq j \leq n$, where $(i, j) \leq (i', j')$ iff $i \leq i'$ and $j \leq j'$. Regarded as a category it may be identified to the category of arrows of Δ^n .

Given $0 \leq i < j < k \leq n$ let

$$(5) \quad a_{i,j,k} : \square \longrightarrow \text{Ar } \Delta^n$$

denote the functor defined as follows:

$$(0, 0) \mapsto (i, j), \quad (0, 1) \mapsto (i, k), \quad (1, 0) \mapsto (j, j), \quad (1, 1) \mapsto (j, k).$$

For any integer $n \geq 0$, we denote by $S_n \mathbf{B}$ the full subcategory of $\mathbf{B}_{\text{Ar } \Delta^n}$ that consists of the objects X satisfying the following two conditions:

- ◊ for any $i \leq n$, $X_{(i,i)}$ is isomorphic to zero in \mathbf{B}_0 ;
- ◊ for any $0 \leq i < j < k \leq n$, $a_{i,j,k}^* X$ is a cocartesian square if $n > 1$.

The definition of $S_n \mathbf{B}$ is similar to that of $S_n \mathcal{C}$, where \mathcal{C} is a Waldhausen category (see [26] for details). Note that $S_0 \mathbf{B}$ is the full subcategory of zero objects in \mathbf{B}_0 . The category $S_1 \mathbf{B}$ consists of the objects $X \in \mathbf{B}_{\Delta^2}$ with X_0 and X_2 isomorphic to zero.

PROPOSITION 3.1. *Let $n \geq 1$ and let $\ell : \Delta^{n-1} \rightarrow \text{Ar } \Delta^n$ be the map taking j to $(0, j+1)$. Then the functor ℓ^* induces an equivalence of categories $S_n \mathbf{B}$ and $\mathbf{B}_{\Delta^{n-1}}$.*

Proof. The proof breaks into two steps.

I. We need some specification both for left systems of diagram categories and for left pointed dérivateurs.

(a) Suppose \mathbf{B} is a left system of diagram categories. Let us consider the following full subcategory I of $\text{Ar } \Delta^n$

$$\left\{ \begin{array}{c} (0, 1) \longrightarrow (0, 2) \longrightarrow \cdots \longrightarrow (0, n) \\ \downarrow \\ (1, 1) \end{array} \right\} \bigcup_{2 \leq i \leq n} (i, i)$$

as well as the map $g : \Delta^{n-1} \rightarrow I$, $j \mapsto (0, j+1)$. Since g is an open immersion, it follows from the immersion lemma that g possesses the left adjoint $f : I^* \rightarrow \Delta^{n-1*}$, $(0, j) \mapsto j-1$ and $(i, i) \mapsto \star$. Hence f^* is a right adjoint to g^* .

Denote by $\tilde{\mathbf{B}}_I$ the full subcategory of \mathbf{B}_I consisting of the objects $X \in \mathbf{B}_I$ such that $X_{(i,i)}$, $1 \leq i \leq n$, are isomorphic to zero. We claim that f^* and g^* are mutually inverse equivalences between $\mathbf{B}_{\Delta^{n-1}}$ and $\tilde{\mathbf{B}}_I$. Indeed, $f^* A$ is in $\tilde{\mathbf{B}}_I$ for every $A \in \mathbf{B}_{\Delta^{n-1}}$ and $g^* f^* = 1$. On the other hand, the adjunction map $B \rightarrow f^* g^* B$ is an isomorphism for every $B \in \tilde{\mathbf{B}}_I$.

(b) Suppose \mathbf{B} is a left pointed dérivateur. Since g is an open immersion, then the functor g^* possesses a right adjoint g_* . Let us show that g^* and g_* are mutually inverse equivalences between $\mathbf{B}_{\Delta^{n-1}}$ and $\tilde{\mathbf{B}}_I$. Indeed, the adjunction map $g^* g_* \rightarrow 1$ is an isomorphism by Corollary 2.2. Since g is an open immersion, we see that $g_* B$ is in $\tilde{\mathbf{B}}_I$ for all $B \in \mathbf{B}_{\Delta^{n-1}}$ (see the corresponding remarks on p. 16). It immediately follows that the adjunction map $B \rightarrow g_* g^* B$ is an isomorphism for every $B \in \tilde{\mathbf{B}}_I$.

II. Second, let $h : I \longrightarrow \text{Ar } \Delta^n$ be the inclusion. It follows from Propositions 1.9 and 2.5 that for every $A \in \mathbf{B}_I$ the object $h_! A$ makes all visible squares

$$\begin{array}{ccccccc}
(0,0) & \longrightarrow & (0,1) & \longrightarrow & (0,2) & \longrightarrow & \cdots \longrightarrow (0,n-1) \longrightarrow (0,n) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & (1,1) & \longrightarrow & (1,2) & \longrightarrow & \cdots \longrightarrow (1,n-1) \longrightarrow (1,n) \\
& & & & \downarrow & & \downarrow & & \downarrow \\
& & & & (2,2) & \longrightarrow & \cdots \longrightarrow (2,n-1) \longrightarrow (2,n) \\
& & & & & & \downarrow & & \downarrow \\
& & & & & & \vdots & & \vdots \\
& & & & & & \downarrow & & \downarrow \\
& & & & & & (n-1,n-1) & \longrightarrow & (n-1,n) \\
& & & & & & & & \downarrow \\
& & & & & & & & (n,n)
\end{array}$$

in the category $\text{Ar } \Delta^n$ cocartesian. By Propositions 1.10 and 2.6 the same is true for all concatenations of visible squares.

It follows from Propositions 1.4 and 2.1 that $h_! A_{(0,0)}$ is isomorphic to zero. By Corollaries 1.5 and 2.2 the canonical morphism

$$A \longrightarrow h^* h_! A$$

is an isomorphism for all $A \in \mathbf{B}_I$. Let $1 \leq i \leq n$ then for any $A \in \widetilde{\mathbf{B}}_I$

$$0 \simeq A_{(i,i)} \simeq h^* h_! A_{(i,i)} = h_! A_{(i,i)}.$$

Thus $h_!$ takes an object A of $\widetilde{\mathbf{B}}_I$ to one of $S_n \mathbf{B}$. We denote the restriction of $h_!$ to $\widetilde{\mathbf{B}}_I$ by the same letter. To show that

$$h_! : \widetilde{\mathbf{B}}_I \longrightarrow S_n \mathbf{B}$$

is an equivalence, we must check that the adjunction morphism

$$(6) \quad h_! h^* B \longrightarrow B$$

is an isomorphism for any $B \in S_n \mathbf{B}$. By the isomorphism axiom it suffices to prove that this map is an isomorphism at each point $(i, j) \in \text{Ar } \Delta^n$. Obviously, it is so at each $(i, j) \in I \cup (0, 0)$.

Given $1 \leq i < j \leq n$ let us consider the square

$$a_{0,i,j} : \square \longrightarrow \text{Ar } \Delta^n.$$

We denote the restriction of $a_{0,i,j}$ to Γ by α . The map (6) induces the map

$$(7) \quad \alpha^* h_! h^* B \longrightarrow \alpha^* B$$

as well as the map

$$h_! h^* B_{(i,j)} \simeq \underline{\mathrm{Holim}}_{\rightarrow} \alpha^* h_! h^* B \longrightarrow \underline{\mathrm{Holim}}_{\rightarrow} \alpha^* B \simeq B_{(i,j)}.$$

Here we have used Lemmas 1.7 and 2.3. We see that the latter map is an isomorphism whenever (7) is. Since $\mathrm{Im} \alpha \subset I$, it follows that (7) is always an isomorphism. Thus (6) is an isomorphism at each $(i, j) \in \mathrm{Ar} \Delta^n$, and hence $h_!$ and h^* are mutual inverses by the isomorphism axiom.

Since $\ell = hg$ the functor

$$\ell^* = g^* h^* : S_n \mathbf{B} \longrightarrow \mathbf{B}_{\Delta^{n-1}}$$

is an equivalence because both g^* and h^* are so by above. If \mathbf{B} is a left system of diagram categories (respectively a left pointed dérivateur), a quasi-inverse to ℓ^* is given by $h_! f^*$ (respectively by $h_! g_*$). This yields the claim. \square

We denote by $\mathbf{B}(I)$ the left system of diagram categories or the left pointed dérivateur respectively defined as $\mathbf{B}(I)_J = \mathbf{B}_{I \times J}$. Every map $f : I \longrightarrow J$ yields a functor $f^* : \mathbf{B}(J) \longrightarrow \mathbf{B}(I)$. Below we shall need the following.

PROPOSITION 3.2. *The structure functor $f^* : \mathbf{B}(J)_0 = \mathbf{B}_J \longrightarrow \mathbf{B}(I)_0 = \mathbf{B}_I$ respects coproducts.*

Proof. By definition, the coproduct of two objects $A, B \in \mathbf{B}_J$ is the value of the functor

$$\mathbf{B}_J \times \mathbf{B}_J \simeq \mathbf{B}_{J \amalg J} \xrightarrow{p_!} \mathbf{B}_J$$

on (A, B) , where $p : J \amalg J \longrightarrow J$ is the canonical map. Propositions 1.8 and 2.4 now imply the assertion. \square

By Propositions 1.8 and 2.4 $f^* : \mathbf{B}(J) \longrightarrow \mathbf{B}(I)$ respects cocartesian squares. Therefore one obtains a functor (denote it by the same letter)

$$f^* : S_n \mathbf{B}(J) \longrightarrow S_n \mathbf{B}(I),$$

and for any bimorphism $\varphi : f \longrightarrow g$, the bimorphism φ^* induces a natural transformation of functors

$$S_n \mathbf{B}(J) \begin{array}{c} \xrightarrow{f^*} \\ \xrightarrow{g^*} \end{array} S_n \mathbf{B}(I).$$

We put $\mathbf{S}_n \mathbf{B}_I = S_n \mathbf{B}(I)$. Then $\mathbf{S}_n \mathbf{B}$ is a presystem of diagram categories or a prédérivateur respectively. $\mathbf{S}_0 \mathbf{B}$ is trivial and for $n \geq 1$ Proposition 3.1 implies an equivalence

$$\mathbf{S}_n \mathbf{B} \simeq \mathbf{B}(\Delta^{n-1}).$$

Since $\mathbf{B}(\Delta^{n-1})$ is a left system of diagram categories or a left pointed dérivateur respectively, it follows that $\mathbf{S}_n \mathbf{B}$ is so as well. We thus obtain a simplicial left system of diagram categories (respectively a left pointed dérivateur)

$$\mathbf{S} \cdot \mathbf{B} : \Delta^n \longmapsto \mathbf{S}_n \mathbf{B}.$$

Consider the following simplicial category:

$$S \cdot \mathbf{B} : \Delta^n \longmapsto S_n \mathbf{B}.$$

For any $n \geq 0$, let $iS_n\mathbf{B}$ denote the subcategory of $S_n\mathbf{B}$ whose objects are those of $S_n\mathbf{B}$ and whose morphisms are isomorphisms in $S_n\mathbf{B}$, and let $i.S_n\mathbf{B}$ be the nerve of $iS_n\mathbf{B}$. We obtain then the following bisimplicial object:

$$i.S. : \Delta^m \times \Delta^n \mapsto i_m S_n \mathbf{B}.$$

LEMMA 3.3. *The space $|i.S.\mathbf{B}|$ is connected.*

Proof. The geometric realization of a bisimplicial set is the diagonal. If $O_1, O_2 \in i_0 S_0 \mathbf{B}$ and $f : O_1 \rightarrow O_2$ is the unique arrow in $S_0 \mathbf{B}$ connecting them, then $A = \sigma_0(f) \in i_1 S_1 \mathbf{B}$ has $\partial_0 A = O_2, \partial_1 A = O_1$. \square

DEFINITION. The *Grothendieck group* $K_0(\mathbf{B})$ is the group generated by the set of isomorphism classes $[B]$ of objects of \mathbf{B}_0 with the relations that $[B] = [A] \cdot [C]$ for every $E \in S_2 \mathbf{B}$ such that $E_{(0,1)} = A, E_{(0,2)} = B$, and $E_{(1,2)} = C$.

LEMMA 3.4. $\pi_1|i.S.\mathbf{B}| \simeq K_0(\mathbf{B})$.

Proof. $\pi_1|i.S.\mathbf{B}|$ is the free group on $\pi_0|i.S_1\mathbf{B}|$ modulo the relations $d_1(x) = d_2(x)d_0(x)$ for every $x \in \pi_0|i.S_2\mathbf{B}|$. This follows from the fact that $\pi_1|i.S_0\mathbf{B}| = 0$ and the Bousfield-Friedlander homotopy spectral sequence [1]. We have that $\pi_0|i.S_1\mathbf{B}|$ is the set of isomorphism classes of objects in \mathbf{B}_0 , $\pi_0|i.S_2\mathbf{B}|$ is the set of isomorphism classes of objects in $S_2\mathbf{B}$, and the maps $d_i : S_2\mathbf{B} \rightarrow S_1\mathbf{B}$ send E to $E_{(1,2)}, E_{(0,2)}$ and $E_{(0,1)}$, respectively. \square

Let \mathcal{A} be an exact category. Its bounded derived category $D^b(\mathcal{A})$ is constructed as follows (we follow here Keller's definition [11]).

Let $H^b(\mathcal{A})$ be the homotopy category of the category of bounded complexes $\mathcal{C} = C^b(\mathcal{A})$, i.e., the quotient category of \mathcal{C} modulo homotopy equivalence. Let $Ac(\mathcal{A})$ denote the full subcategory of $H^b(\mathcal{A})$ consisting of acyclic complexes. A complex

$$X^n \rightarrow X^{n+1} \rightarrow X^{n+2}$$

is called *acyclic* if each map $X^n \rightarrow X^{n+1}$ decomposes in \mathcal{A} as $X^n \xrightarrow{e_n} D^n \xrightarrow{m_n} X^{n+1}$ where e_n is an (admissible) epimorphism and m_n is an (admissible) monomorphism; furthermore, $D^n \xrightarrow{m_n} X^{n+1} \xrightarrow{e_{n+1}} D^{n+1}$ must be an exact sequence.

If an exact category is idempotent complete then every contractible complex is acyclic. Denote by $\mathcal{N} = \mathcal{N}_{\mathcal{A}}$ the full subcategory of $H^b(\mathcal{A})$ whose objects are the complexes isomorphic in $H^b(\mathcal{A})$ to acyclic complexes. There is another description of \mathcal{N} . Let $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ be the universal additive functor to an idempotent complete exact category $\tilde{\mathcal{A}}$. It is exact and reflects exactness, and \mathcal{A} is closed under extensions in $\tilde{\mathcal{A}}$ (see [20, A.9.1]). Then a complex with entries in \mathcal{A} belongs to \mathcal{N} iff its image in $H^b(\tilde{\mathcal{A}})$ is acyclic. The category $\mathcal{N}_{\tilde{\mathcal{A}}} = Ac(\tilde{\mathcal{A}})$ is a thick subcategory in $H^b(\tilde{\mathcal{A}})$. Note that an object over $\tilde{\mathcal{A}}$ is acyclic iff it has trivial homology computed in an ambient abelian category. It follows that \mathcal{N} is a thick subcategory in $H^b(\mathcal{A})$. Denote by Σ the multiplicative system associated to \mathcal{N} and call the elements of Σ *quasi-isomorphisms*. A map s is a quasi-isomorphism iff in any triangle

$$L \xrightarrow{s} M \rightarrow N \rightarrow L[1]$$

the complex N belongs to \mathcal{N} .

The derived category is defined as

$$D^b(\mathcal{A}) = H^b(\mathcal{A})/\mathcal{N} = H^b(\mathcal{A})[\Sigma^{-1}].$$

Clearly, a map is a quasi-isomorphism iff its image in $C^b(\tilde{\mathcal{A}})$ is a quasi-isomorphism and iff its image in $D^b(\mathcal{A})$ is an isomorphism.

Recall that the Grothendieck group $K_0(D^b(\mathcal{A}))$ is defined as the group generated by the set of isomorphism classes $[B]$ of objects of $D^b(\mathcal{A})$ with the relations that $[B] = [A] + [C]$ for every triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

According to [11] (consult also [4]) the hyperfunctor

$$I \longmapsto D^b(\mathcal{A}^I)$$

yields a pointed bidérivateur of the domain \mathcal{Dirf} . It will be denoted by $\mathbf{D}^b(\mathcal{A})$.

LEMMA 3.5. $K_0(\mathbf{D}^b(\mathcal{A})) = K_0(D^b(\mathcal{A}))$.

Proof. It is enough to observe that there is a 1-1 correspondence between the isomorphism classes of objects in $S_2\mathbf{D}^b(\mathcal{A})$ and the isomorphism classes of triangles in $D^b(\mathcal{A})$ (consult [4, 13] for details). \square

DEFINITION. The *Algebraic K-theory* for a small left system of diagram categories of the domain \mathcal{Ord}^* or for a left pointed dérivateur \mathbf{B} of the domain \mathcal{Dia} is given by the pointed space (a fixed zero object 0 of \mathbf{B}_0 is taken as a basepoint)

$$K(\mathbf{B}) = \Omega|i.S.\mathbf{B}|.$$

The K -groups of \mathbf{B} are the homotopy groups of $K(\mathbf{B})$

$$K_*(\mathbf{B}) = \pi_*(\Omega|i.S.\mathbf{B}|) = \pi_{*+1}(|i.S.\mathbf{B}|).$$

CONVENTION. We shall also denote by $0 \in \mathbf{B}_I$ the object $const^*0$ where $const : I \longrightarrow 0$ is the constant map and 0 is the fixed zero object of \mathbf{B}_0 . Let $(L.s.d.c., \text{Left pointed dérivateurs})$ denote the corresponding categories of left systems of diagram categories and left pointed dérivateurs and right exact functors. In order to make the map

$$(L.s.d.c., \text{Left pointed dérivateurs}) \xrightarrow{K} (Spaces)$$

functorial, in what follows we assume that $\iota_{F,f} : f^*F \longrightarrow Ff^*$ are identities for any right exact morphism $F : \mathbf{A} \longrightarrow \mathbf{B}$ and any map f in \mathcal{Dia} .

Any right exact functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ induces a map $F_* : K(\mathbf{A}) \longrightarrow K(\mathbf{B})$ of spaces and of their homotopy groups $K_i(\mathbf{A}) \longrightarrow K_i(\mathbf{B})$.

We can apply the S -construction to each $\mathbf{S}_n\mathbf{B}$, obtaining a bisimplicial left system of diagram categories or a bisimplicial left pointed dérivateur respectively. Iterating this construction, we can form the multisimplicial object $\mathbf{S}^n\mathbf{B} = \mathbf{S}.\mathbf{S} \cdots \mathbf{S}.\mathbf{B}$ and the multisimplicial categories $iS.^n\mathbf{B}$ of isomorphisms. If the additivity theorem holds, we show that $|iS.^n\mathbf{B}|$ is the loop space of $|iS.^{n+1}\mathbf{B}|$ for any $n \geq 1$ and that the sequence

$$\Omega|i.S.\mathbf{B}|, \Omega|i.S.S.\mathbf{B}|, \dots, \Omega|i.S.^n\mathbf{B}|, \dots$$

forms a connective Ω -spectrum \mathbf{KB} . In this case, one can think of the K -theory of \mathbf{B} in terms of this spectrum. This does not affect the K -groups, because:

$$\pi_i(\mathbf{KB}) = \pi_i(K(\mathbf{B})) = K_i(\mathbf{B}), \quad i \geq 0.$$

The additivity theorem remains open for $K(\mathbf{B})$. Nevertheless, if the definition of K -theory as $\Omega|i.S.\mathbf{B}|$ is substituted with the infinite loop space

$$\Omega^\infty|i.S.^\infty\mathbf{B}| = \lim_n \Omega^n|i.S.^n\mathbf{B}|,$$

then the additivity theorem does hold (excluding pathological cases we never have in practice).

Maltsiniotis [14] uses the Q -construction to define a K -theory space of a triangulated dérivateur. This construction can be extended to arbitrary left systems of diagram categories or left pointed dérivateurs if we replace bicartesian squares in Maltsiniotis' definition by cocartesian squares. To be more precise, it is given by the bisimplicial category $Q\mathbf{B} = \{Q_{m,n}\mathbf{B}\}_{m,n \geq 0}$ with $Q_{m,n}\mathbf{B}$ being the full subcategory in $\mathbf{B}_{\Delta^m \times \Delta^n}$ such that every $X \in Q_{m,n}\mathbf{B}$ makes any square $i : \square \longrightarrow \Delta^m \times \Delta^n$ cocartesian. Let $iQ\mathbf{B}$ denote the corresponding maximal groupoid. Then $iQ\mathbf{B}$ is a trisimplicial object and the K -theory space is defined as $\Omega|\text{diag}(iQ\mathbf{B})|$.

According to [5] the resulting K -theory is equivalent to that defined by us in terms of the S -construction. The proof is based on [26, p. 334] and makes sense without any problem to our setting.

4. SIMPLICIAL PRELIMINARIES

Multi-simplicial sets will naturally arise in this work. It will be important that we can work with them directly, without diagonalizing away all the structure. Such work depends on a couple of lemmas which we give below. We formulate them for bisimplicial sets as the corresponding lemmas for multi-simplicial sets are immediate consequences, by taking suitable diagonals.

LEMMA 4.1. *Let $X.. \longrightarrow Y..$ be a map of bisimplicial sets. Suppose that for every n , the map $X._n \longrightarrow Y._n$ is a homotopy equivalence. Then $X.. \longrightarrow Y..$ is a homotopy equivalence.*

Proof. See [19]. □

LEMMA 4.2. *Let $X.. \longrightarrow Y.. \longrightarrow Z..$ be a sequence of bisimplicial sets so that $X.. \longrightarrow Z..$ is constant. Suppose that $X._n \longrightarrow Y._n \longrightarrow Z._n$ is a fibration up to homotopy, for every n . Suppose further that $Z._n$ is connected for every n . Then $X.. \longrightarrow Y.. \longrightarrow Z..$ is a fibration up to homotopy.*

Proof. [25, Lemma 5.2]. □

LEMMA 4.3. *Let \mathcal{A} and \mathcal{B} be two small simplicial categories so that the underlying sets of objects form simplicial sets and let $i\mathcal{A}$ and $i\mathcal{B}$ denote the corresponding simplicial subcategories of isomorphisms. Then every equivalence $F : \mathcal{A} \longrightarrow \mathcal{B}$ induces a homotopy equivalence of bisimplicial objects $F : i.\mathcal{A} \longrightarrow i.\mathcal{B}$. In particular, if \mathcal{A} and \mathcal{B} happen to be two left systems of diagram categories or two left pointed*

dérivateurs, then every right exact equivalence $F : \mathcal{A} \longrightarrow \mathcal{B}$ induces a homotopy equivalence $F : i.S.\mathcal{A} \longrightarrow i.S.\mathcal{B}$.

Proof. By Lemma 4.1 it suffices to show that each $F_n : i.\mathcal{A}_n \longrightarrow i.\mathcal{B}_n$ is a homotopy equivalence. The latter is obvious, because $F_n : i\mathcal{A}_n \longrightarrow i\mathcal{B}_n$ is an equivalence of categories.

If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an equivalence of left systems of diagram categories or left pointed dérivateurs, it is directly verified that F induces an equivalence of categories $F : S_n\mathcal{A} \longrightarrow S_n\mathcal{B}$, and hence a homotopy equivalence $F : i.S.\mathcal{A} \longrightarrow i.S.\mathcal{B}$. \square

Let C and D be two simplicial objects in a category \mathcal{C} and let Δ/Δ^1 denote the category of objects over Δ^1 in Δ ; the objects are the maps $\Delta^n \longrightarrow \Delta^1$. For any simplicial object C in \mathcal{C} let C^* denote the composed functor

$$\begin{aligned} (\Delta/\Delta^1)^{\text{op}} &\longrightarrow \Delta^{\text{op}} \xrightarrow{C} \mathcal{C} \\ (\Delta^n \longrightarrow \Delta^1) &\longmapsto \Delta^n \longmapsto C_n. \end{aligned}$$

Then a *simplicial homotopy* of maps from C to D is a natural transformation $C^* \longrightarrow D^*$ [26, p. 335].

There is a functor $P : \Delta \longrightarrow \Delta$ with $P\Delta^n = \Delta^{n+1}$ such that the natural map $s_0 : \Delta^n \longrightarrow \Delta^{n+1} = P\Delta^n$ is a natural transformation $\text{id}_\Delta \longrightarrow P$. It is obtained by formally adding an initial object $0'$ to each Δ^n and then identifying $\{0' < 1 < \dots < n\}$ with Δ^{n+1} . Thus $P(s_i) = s_{i+1}$ and $P(d_i) = d_{i+1}$. If A is a simplicial object in \mathcal{A} , the *path space* PA is the simplicial object obtained by composing A with P . Thus $PA_n = A_{n+1}$, and the i th face operator on PA is the ∂_{i+1} of A , and the i th degeneracy operator on PA is the σ_{i+1} of A . Moreover, the maps $\partial_0 : A_{n+1} \longrightarrow A_n$ form a simplicial map $PA \longrightarrow A$.

Let us write A_0 for the constant simplicial object at A_0 . The natural maps $\sigma_0^{n+1} : A_0 \longrightarrow A_{n+1}$ form a simplicial map $\iota : A_0 \longrightarrow PA$, and the maps $A_{n+1} \longrightarrow A_0$ induced by the canonical inclusion of $\Delta^0 = \{0\}$ in Δ^{n+1} form a simplicial map $\rho : PA \longrightarrow A_0$ such that $\rho\iota$ is the identity on A_0 . On the other hand, $\iota\rho$ is homotopic to the identity on PA . The homotopy is given by the natural transformation

$$(a : \Delta^n \longrightarrow \Delta^1) \longmapsto (\varphi_a^* : A_{n+1} \longrightarrow A_{n+1})$$

induced from $(a : \Delta^n \longrightarrow \Delta^1) \longmapsto (\varphi_a : \Delta^{n+1} \longrightarrow \Delta^{n+1})$ where $\varphi_a(0) = 0$ and

$$\varphi_a(j+1) = \begin{cases} j+1, & a(j) = 1 \\ 0, & a(j) = 0. \end{cases}$$

This shows that PA is homotopy equivalent to A_0 .

5. Γ -SPACES

In this section we use Segal's machine [19] to get some information about the K -theory $K(\mathbf{B})$. We start with preparations.

Given a finite set T by $\mathcal{P}(T)$ denote the set of subsets of T and the set $\{1, 2, \dots, n\}$ is denoted by \mathbf{n} .

DEFINITION. I. Γ is the category whose objects are all finite sets, and whose morphisms from S to T are the maps $\theta : S \longrightarrow \mathcal{P}(T)$ such that $\theta(\alpha)$ and $\theta(\beta)$ are disjoint when $\alpha \neq \beta$. The composite of $\theta : S \longrightarrow \mathcal{P}(T)$ and $\varphi : T \longrightarrow \mathcal{P}(U)$ is $\psi : S \longrightarrow \mathcal{P}(U)$, where $\psi(\alpha) = \bigcup_{\beta \in \theta(\alpha)} \varphi(\beta)$.

II. A Γ -space is a contravariant functor A from Γ to spaces such that

- (a) $A(\mathbf{0})$ is contractible, and
- (b) for any n the map $p_n : A(\mathbf{n}) \longrightarrow A(\mathbf{1}) \times \cdots \times A(\mathbf{1})$ induced by the maps $i_k : \mathbf{1} \longrightarrow \mathbf{n}$ in Γ , where $i_k(\mathbf{1}) = \{k\} \subset \mathbf{n}$, is a homotopy equivalence.

We shall refer to $A(\mathbf{1})$ as the *underlying space*.

There is a covariant functor $\Delta \longrightarrow \Gamma$ which takes Δ^m to \mathbf{m} and $f : \Delta^m \longrightarrow \Delta^n$ to $\theta(i) = \{j \in \mathbf{n} \mid f(i-1) < j \leq f(i)\}$. Using this functor one can regard Γ -spaces as simplicial spaces.

Segal uses a realization functor $A \longrightarrow |A|$ for simplicial spaces which is slightly different from the usual one (see [19, Appendix A]). If A is a Γ -space its realization will mean the realization of the simplicial space it defines.

DEFINITION. If A is a Γ -space, its classifying space is the Γ -space BA such that, for any finite set S , $BA(S)$ is the realization of the Γ -space $T \longmapsto A(S \times T)$.

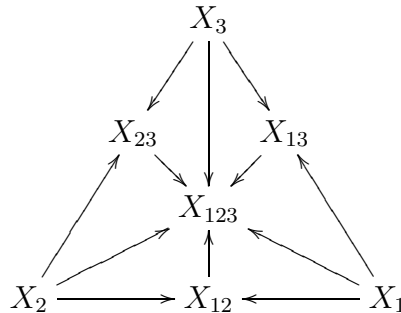
If A is a Γ -space the spaces $A(\mathbf{1}), BA(\mathbf{1}), B^2A(\mathbf{1}), \dots$ form a spectrum, denoted by \mathbf{BA} . The reason of introducing Γ -spaces is that they arise naturally from categories.

DEFINITION. A Γ -category is a contravariant functor \mathcal{C} from Γ to categories such that

- (a) $\mathcal{C}(\mathbf{0})$ is equivalent to the category with one object and one morphism;
- (b) for each n the functor $p_n : \mathcal{C}(\mathbf{n}) \longrightarrow \mathcal{C}(\mathbf{1}) \times \cdots \times \mathcal{C}(\mathbf{1})$ induced by the maps $i_k : \mathbf{1} \longrightarrow \mathbf{n}$ in Γ is an equivalence of categories.

If \mathcal{C} is a Γ -category, $|\mathcal{C}|$ is a Γ -space. Here $|\mathcal{C}|$ means the functor $S \longmapsto |\mathcal{C}(S)|$.

Γ -categories arise in the following way. Let \mathcal{C} be a category with a zero object 0 in which sums exist. If S is a finite set, let $\mathcal{P}(S)$ denote the category of subsets of S and their inclusions — this should not cause confusion with the earlier use of $\mathcal{P}(S)$. Let $\mathcal{C}(S)$ denote the category whose objects are the functors from $\mathcal{P}(S)$ to which take disjoint unions to sums. For example, $\mathcal{C}(\mathbf{3})$ is the category of diagrams in \mathcal{C} of the form



in which each straight line (such as $X_1 \longrightarrow X_{123} \longleftarrow X_{23}$) is an expression of the middle object as a sum of ends. The morphisms of Γ were so defined that the morphisms from S to T in Γ correspond precisely to functors from $\mathcal{P}(S)$ to

$\mathcal{P}(T)$ which preserve disjoint unions. The category $i\mathcal{C}(S)$ of isomorphisms in $\mathcal{C}(S)$ satisfies the definition above because, for example, the forgetful functor $i\mathcal{C}(\mathbf{3}) \longrightarrow i\mathcal{C} \times i\mathcal{C} \times i\mathcal{C}$, which takes the diagram above to (X_1, X_2, X_3) is an equivalence of categories.

Conforming to the terminology and notation of [26, section 1.8], we denote the resulting simplicial category by $N\mathcal{C}$. By definition, $N_0\mathcal{C} = 0$ and $N_n\mathcal{C} = \mathcal{C}(\mathbf{n})$ for each $n \geq 1$. We refer to the simplicial category $N\mathcal{C}$ as the *nerve with respect to the composition law*. By construction, the space $|i.N\mathcal{C}|$ is $B|i.\mathcal{C}|(\mathbf{1})$. By $N.\mathbf{B}$ will be denoted the nerve with respect to the composition law associated to the category \mathbf{B}_0 .

Observe that any functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ respecting sums yields a map of bisimplicial objects

$$f^* : i.N\mathcal{C} \longrightarrow i.N\mathcal{D}.$$

It follows that for a given left system of diagram categories or a left pointed d rivateur \mathbf{B} one can also produce the multisimplicial categories $iN.^mS.^n\mathbf{B}$, $m, n \geq 0$, and the spaces $|i.N.^mS.^n\mathbf{B}|$ by iterating the N - and S -constructions.

PROPOSITION 5.1. *$|i.S.\mathbf{B}|$ is canonically an infinite loop space, and hence is so the K -theory space $K(\mathbf{B})$.*

Proof. The above considerations show that $|i.S.\mathbf{B}|$ is the underlying space of a Γ -space, with respect to the composition law produced by coproduct. \square

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a right exact functor between left systems of diagram categories or left pointed d rivateurs, respectively. Let further $N_n(\mathbf{A} \longrightarrow \mathbf{B})$ denote the fibred product of the diagram

$$N_n\mathbf{A} \xrightarrow{F} N_n\mathbf{B} \xleftarrow{\partial_0} (PN.\mathbf{B})_n = N_{n+1}\mathbf{B}.$$

An object of $N_n(\mathbf{A} \longrightarrow \mathbf{B})$ is a triple (A, c, B) with $A \in N_n\mathbf{A}$, $B \in N_{n+1}\mathbf{B}$, $c : F(A) \longrightarrow \partial_0(B)$ an isomorphism in $N_n\mathbf{B}$. One obtains a simplicial category

$$N.(\mathbf{A} \longrightarrow \mathbf{B}) : \Delta^n \longmapsto N_n(\mathbf{A} \longrightarrow \mathbf{B}).$$

For every n , there is a functor

$$g : \mathbf{B}_0 = N_1\mathbf{B} \longrightarrow N_n(\mathbf{A} \longrightarrow \mathbf{B})$$

defined by $B \longmapsto (0, 1, v^*B)$ with $v : \Delta^{n+1} \longrightarrow \Delta^1$, $i \longmapsto 0$ if $i = 0$ and $i \longmapsto 1$ otherwise.

Regarding \mathbf{B}_0 as a trivial simplicial category, we obtain a sequence

$$\mathbf{B}_0 \xrightarrow{g} N.(\mathbf{A} \longrightarrow \mathbf{B}) \xrightarrow{p} N.\mathbf{A}$$

where p is the projection. The latter sequence yields the sequence

$$(8) \quad i.S.\mathbf{B} \xrightarrow{g} i.N.S.(\mathbf{A} \longrightarrow \mathbf{B}) \xrightarrow{p} i.N.S.\mathbf{A}$$

with $N.S.(\mathbf{A} \longrightarrow \mathbf{B}) = N.(S.\mathbf{A} \longrightarrow S.\mathbf{B})$. Note that the space $|i.N.S.\mathbf{A}|$ is $B|i.S.\mathbf{A}|(\mathbf{1})$, where $B|i.S.\mathbf{A}|$ is the Γ -space associated to $|i.S.\mathbf{A}|$.

LEMMA 5.2. *The sequence (8) is a fibration up to homotopy.*

Proof. By Lemma 4.2 it is enough to show that for every n the sequence $i.S.\mathbf{B} \longrightarrow i.N_n S.(\mathbf{A} \longrightarrow \mathbf{B}) \longrightarrow i.N_n S.\mathbf{A}$ is a fibration (since the base term $i.N_n S.\mathbf{A} = i.\text{Hom}(\mathcal{P}(\mathbf{n}), S.\mathbf{A}) \simeq (i.S.\mathbf{A})^n$ is connected for every n by Lemma 3.3). We will show that the sequence is the same, up to homotopy, as the trivial fibration sequence associated to the product $i.S.\mathbf{B} \times i.N_n S.\mathbf{A}$.

Let $u : \Delta^1 \longrightarrow \Delta^{n+1}$ be the map $0;1 \longmapsto 0;1$. Also, consider the maps $d_0 : \Delta^n \longrightarrow \Delta^{n+1}$ and $s_0 : \Delta^{n+1} \longrightarrow \Delta^n$. We construct the following diagram for any $B \in N_{n+1}\mathbf{B}$,

$$B' = v^* u^* B \xrightarrow{\varphi} B \xleftarrow{\psi} B'' = s_0^* \partial_0 B.$$

For any subset S of $[\mathbf{n} + 1]$,

$$B'_S = \begin{cases} B_1, & 1 \in S \\ 0, & 1 \notin S \end{cases}$$

and

$$B''_S = \begin{cases} B_{S \setminus \{1\}}, & 1 \in S \\ B_S, & 1 \notin S \end{cases}$$

whence the definitions of φ and ψ follow. Note that $B'_S \xrightarrow{\varphi_S} B_S \xleftarrow{\psi_S} B''_S$ belongs to $N_2\mathbf{B}$.

The map $N_n(\mathbf{A} \longrightarrow \mathbf{B}) \longrightarrow N_n\mathbf{A} \times \mathbf{B}_0$, $(A, c, B) \longmapsto (A, B_{\{1\}})$, is an equivalence of categories. A quasi-inverse is given by the functor

$$(A, B) \longmapsto (A, 1, s_0^* F A \coprod v^* B).$$

Thus the induced map $i.N_n S.(\mathbf{A} \longrightarrow \mathbf{B}) \longrightarrow i.N_n S.\mathbf{A} \times i.S.\mathbf{B}$ is a homotopy equivalence by Lemma 4.1.

This homotopy equivalence fits into the following commutative diagram

$$\begin{array}{ccccc} i.S.\mathbf{B} & \longrightarrow & i.N_n S.(\mathbf{A} \longrightarrow \mathbf{B}) & \longrightarrow & i.N_n S.\mathbf{A} \\ \downarrow 1 & & \downarrow & & \downarrow 1 \\ i.S.\mathbf{B} & \longrightarrow & i.N_n S.\mathbf{A} \times i.S.\mathbf{B} & \longrightarrow & i.N_n S.\mathbf{A} \end{array}$$

Being homotopy equivalent to the trivial fibration (the lower row of the diagram), we conclude that the upper sequence is a fibration, as was to be shown. \square

As above, one can construct the sequence

$$i.\mathbf{B}_0 \longrightarrow P(i.N.\mathbf{B}) \longrightarrow i.N.\mathbf{B}.$$

The composite map is constant and the middle term is contractible, so we obtain a map well defined up to homotopy,

$$|i.\mathbf{B}_0| \longrightarrow \Omega|i.N.\mathbf{B}|.$$

By naturality we can substitute \mathbf{B} with the simplicial category $S.\mathbf{B}$ in the above sequence. We obtain a sequence

$$i.S.\mathbf{B} \longrightarrow P(i.N.S.\mathbf{B}) \longrightarrow i.N.S.\mathbf{B}$$

where the “ P ” refers to the N -direction. It follows from the preceding lemma that the sequence is a fibration up to homotopy. Thus $|i.S.\mathbf{B}| \longrightarrow \Omega|i.N.S.\mathbf{B}|$ is a

homotopy equivalence and more generally therefore, in view of Lemma 4.1, also the map $|i.N.^n S.B| \longrightarrow \Omega|i.N.^{n+1} S.B|$. There results a spectrum

$$n \longmapsto |i.N.^n S.B|,$$

which is actually a Ω -spectrum. It is nothing but the spectrum $n \longmapsto B^n|i.S.B|(1)$ produced by Segal's machine.

As in the sequence

$$|i.S.B| \longrightarrow \Omega|i.N.S.B| \longrightarrow \Omega\Omega|i.N.N.S.B| \longrightarrow \dots$$

all the maps are homotopy equivalences, then so is the map

$$|i.S.B| \longrightarrow \Omega^\infty|i.N.^\infty S.B| = \lim_n \Omega^n|i.N.^n S.B|$$

COROLLARY 5.3. *Suppose we are given a sequence $\mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{C}$ of right exact morphisms of left systems of diagram categories or left pointed d erivateurs respectively. Then the square*

$$\begin{array}{ccc} i.S.B & \longrightarrow & i.N.S.(\mathbf{A} \longrightarrow \mathbf{B}) \\ \downarrow & & \downarrow \\ i.S.C & \longrightarrow & i.N.S.(\mathbf{A} \longrightarrow \mathbf{C}) \end{array}$$

is homotopy cartesian.

Proof. There is a commutative diagram

$$\begin{array}{ccccc} i.S.B & \longrightarrow & i.N.S.(\mathbf{A} \longrightarrow \mathbf{B}) & \longrightarrow & i.N.S.A \\ \downarrow & & \downarrow & & \downarrow \text{id} \\ i.S.C & \longrightarrow & i.N.S.(\mathbf{A} \longrightarrow \mathbf{C}) & \longrightarrow & i.N.S.A \end{array}$$

in which the rows are fibrations up to homotopy by Lemma 5.2. Therefore the square on the left is homotopy cartesian. \square

COROLLARY 5.4. *The following two assertions are valid.*

(1) *To a right exact morphism there is associated a fibration*

$$i.S.B \longrightarrow i.S.C \longrightarrow i.N.S.(\mathbf{B} \longrightarrow \mathbf{C}).$$

(2) *If \mathbf{C} is a retract of \mathbf{B} (by right exact functors) there is a splitting*

$$i.S.B \simeq i.S.C \times i.N.S.(\mathbf{C} \longrightarrow \mathbf{B}).$$

Proof. (1). If $\mathbf{A} = \mathbf{B}$ the space $|i.N.S.(\mathbf{A} = \mathbf{A})|$ is contractible whence the first assertion.

(2). This is the case of Corollary 5.3 where the composed map $\mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{C}$ is an identity map since $i.N.S.(\mathbf{A} \longrightarrow \mathbf{C})$ is contractible in that case. \square

6. THE ADDITIVITY THEOREM

Let \mathbf{B} be either a left system of diagram categories or a left pointed d  rivateur. Denote by \mathbf{E}_0 the full subcategory in \mathbf{B}_\square consisting of the cocartesian squares $B \in \mathbf{B}_\square$ with $B_{(1,0)}$ isomorphic to zero. If we replace \mathbf{B} by $\mathbf{B}(I)$, we define the category \mathbf{E}_I similar to \mathbf{E}_0 . One obtains a left system of diagram categories or a left pointed d  rivateur \mathbf{E} respectively.

LEMMA 6.1. *The map $l : \Delta^1 \longrightarrow \square$, $i \longmapsto (0, i)$, induces an equivalence of categories $l^* : \mathbf{E}_0 \longrightarrow \mathbf{B}_{\Delta^1}$. It also induces a right exact equivalence $\mathbf{E} \longrightarrow \mathbf{B}(\Delta^1)$.*

Proof. The map l factors as

$$\Delta^1 \xrightarrow{g} \ulcorner \xrightarrow{h} \square.$$

The proof of Proposition 3.1 shows that $l^* : \mathbf{E}_0 \longrightarrow \mathbf{B}_{\Delta^1}$ is an equivalence (for left pointed d  rivateurs use the fact that g is an open immersion). Obviously, the induced morphism $\mathbf{E} \longrightarrow \mathbf{B}(\Delta^1)$ is an equivalence. It is right exact by Propositions 1.8 and 2.4. \square

COROLLARY 6.2. *The map $f : \square \longrightarrow \text{Ar } \Delta^2$, $(i, j) \longmapsto (i, j + 1)$, induces an equivalence of categories $f^* : \mathbf{S}_2\mathbf{B} \longrightarrow \mathbf{E}_0$. It also induces a right exact equivalence $\mathbf{S}_2\mathbf{B} \longrightarrow \mathbf{E}$.*

Proof. Let $\ell : \Delta^1 \longrightarrow \text{Ar } \Delta^2$ be the map $i \longmapsto (0, i + 1)$. It factors as

$$\Delta^1 \xrightarrow{l} \square \xrightarrow{f} \text{Ar } \Delta^2$$

where l is the map of Lemma 6.1. By Proposition 3.1 it follows that $\ell^* = l^*f^* : \mathbf{S}_2\mathbf{B} \longrightarrow \mathbf{B}_{\Delta^1}$ is an equivalence. By Lemma 6.1 l^* is an equivalence, and hence f^* is so. \square

Below we shall need the following.

LEMMA 6.3. *Let \mathbf{B} be either a left system of diagram categories or a left pointed d  rivateur, and let $B \in \mathbf{B}_\square$ be a cocartesian square such that the map $B_{(0,0)} \longrightarrow B_{(0,1)}$ (respectively the map $B_{(0,0)} \longrightarrow B_{(1,0)}$) is an isomorphism. Then $B_{(1,0)} \longrightarrow B_{(1,1)}$ (respectively $B_{(0,1)} \longrightarrow B_{(1,1)}$) is an isomorphism as well. On the other hand, a square with two parallel arrows being isomorphisms is cocartesian.*

Proof. Suppose that the map $B_{(0,0)} \longrightarrow B_{(0,1)}$ is an isomorphism. Let $q : \square \longrightarrow \Delta^1$ denote the functor $(\varepsilon, \eta) \longmapsto \varepsilon$, and let $i : \Delta^1 \longrightarrow \square$ be the functor $\varkappa \longmapsto (\varkappa, 0)$. Then i is a left adjoint to q and hence $i_! \simeq q^*$. The map i factors as

$$\Delta^1 \xrightarrow{l} \ulcorner \xrightarrow{i_r} \square$$

where $l(\varkappa) = (\varkappa, 0)$. By Propositions 1.9 and 2.5 the object $i_!i^*B \simeq i_{r!}(l^*i^*B)$ is cocartesian.

Let $\beta : (i_!i^*B \simeq)q^*i^*B \longrightarrow B$ be the adjunction morphism. Then $\beta_{(0,0)} = \beta_{(1,0)} = 1$ and $\beta_{(0,1)}$ is an isomorphism by assumption. It follows that $i_r^*\beta$ is an isomorphism.

Consider the following commutative square:

$$\begin{array}{ccc} i_{r!}i_r^*i_{!}i^*B & \xrightarrow{i_{r!}i_r^*\beta} & i_{r!}i_r^*B \\ \downarrow & & \downarrow \\ i_{!}i^*B & \xrightarrow{\beta} & B \end{array}$$

The upper arrow is an isomorphism. The vertical maps are isomorphisms too, because both B and $i_{!}i^*B$ are cocartesian. We see that β is an isomorphism as well. This implies that $B_{(1,0)} \rightarrow B_{(1,1)} \simeq \beta_{(1,1)}$ is an isomorphism. The corresponding assertion when $B_{(0,0)} \rightarrow B_{(1,0)}$ is an isomorphism is deduced from the first one by applying the autoequivalence $\tau : \square \rightarrow \square$ transposing the vertices $(1,0)$ and $(0,1)$.

On the other hand, if both $B_{(0,0)} \rightarrow B_{(0,1)}$ and $B_{(1,0)} \rightarrow B_{(1,1)}$ are isomorphisms, it follows that $\beta : i_{!}i^*B \rightarrow B$ is an isomorphism. Since $i_{!}i^*B$ is cocartesian, it follows that B is cocartesian, too. \square

We want to construct a map $\alpha : \mathbf{B}_0 \rightarrow \mathbf{E}_0$ that takes an object $A \in \mathbf{B}_0$ to one in \mathbf{E}_0 which is depicted in \mathbf{B}_0 as the square

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0. \end{array}$$

Let $s_0 : \Delta^1 \rightarrow 0$ be the unique map. First, suppose that \mathbf{B} is a left system of diagram categories and $d : \square^* \rightarrow \Delta^{1*}$ is the map $(0,i) \mapsto i$ and $(1,i) \mapsto \star$. We put then $\alpha = d^*s_0^*$. If \mathbf{B} is a left pointed d  rivateur, let $l : \Delta^1 \rightarrow \square$ be the map $i \mapsto (0,i)$. Then l is an open immersion, and hence there is a right adjoint functor l_* to l^* . In this case $\alpha = l_*s_0^*$.

Let $j : \square \rightarrow \Delta^1$ be the map $(\varepsilon, \eta) \mapsto \eta$. The morphism j^* takes $B \in \mathbf{B}_{\Delta^1}$ to a square in \mathbf{B}_{\square} which is depicted in \mathbf{B}_0 as

$$\begin{array}{ccc} B_0 & \longrightarrow & B_1 \\ 1 \downarrow & & \downarrow 1 \\ B_0 & \longrightarrow & B_1. \end{array}$$

Let \mathbf{B} be a left system of diagram categories and let $u : \Delta^{1*} \rightarrow 0^*$ be the map $0 \mapsto \star$ and $1 \mapsto 0$. Then u^* takes an object $B \in \mathbf{B}_0$ to one in \mathbf{B}_{Δ^1} with $u^*B_0 = 0$ and $u^*B_1 = B$. We put $\beta = j^*u^*$. In turn, if \mathbf{B} is a left pointed d  rivateur, consider the map $v : 0 \rightarrow \Delta^1$ with $v(0) = 1$. Then $v_{!}B_0 = 0$ and $v_{!}B_1 = B$ for any $B \in \mathbf{B}_0$. In this case $\beta := j^*v_{!}$.

The β takes an object $B \in \mathbf{B}_0$ to the square

$$\begin{array}{ccc} 0 & \longrightarrow & B \\ \downarrow & & \downarrow 1 \\ 0 & \longrightarrow & B \end{array}$$

Let \mathbf{B} be either a left system of diagram categories or a left pointed d  rivateur, and let \mathbf{B}' and \mathbf{B}'' be either two left subsystems of diagram categories or two left pointed subd  rivateurs respectively in such a way that the inclusion morphisms are right exact. There are three natural right exact morphisms $s, t, q : \mathbf{E} \longrightarrow \mathbf{B}$ taking an object $E \in \mathbf{E}$ to $E_{(0,0)}$, $E_{(0,1)}$, and $E_{(1,1)}$ respectively. Define $\mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'')$ as a presystem of diagram categories or as a pred  rivateur respectively that consists of the squares $E \in \mathbf{E}$ with $E_{(0,0)} \in \mathbf{B}'$ and $E_{(1,1)} \in \mathbf{B}''$. Then $\mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'')$ is a left system of diagram categories or a left pointed d  rivateur respectively. For $\mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'')$ is equivalent to the fibred product of the diagram $\mathbf{E} \xrightarrow{(s,q)} \mathbf{B} \times \mathbf{B} \longleftarrow \mathbf{B}' \times \mathbf{B}''$. We note that $\mathbf{E} = \mathbf{E}(\mathbf{B}, \mathbf{B}, \mathbf{B})$.

In order to have the unital and associative H -space structure to $|i.S.\mathbf{B}|$ induced by coproduct \amalg via the map

$$|i.S.\mathbf{B}| \times |i.S.\mathbf{B}| \xrightarrow{\sim} |i.S.\mathbf{B} \times i.S.\mathbf{B}| \xrightarrow{\amalg} |i.S.\mathbf{B}|,$$

we must have good choices for $A \amalg B$, $A, B \in S_n \mathbf{B}$, in such a way that $f^*(A \amalg B) = f^*(A) \amalg f^*(B)$ where $f : \Delta^m \longrightarrow \Delta^n$ is a structure map in Δ (one always has an isomorphism between them because f^* respects coproducts by Lemma 3.2). We would have then a simplicial equivalence $\amalg(\amalg \times 1) \simeq \amalg(1 \times \amalg)$

$$\begin{array}{ccc} i.S.\mathbf{B} \times i.S.\mathbf{B} \times i.S.\mathbf{B} & \xrightarrow{\amalg \times 1} & i.S.\mathbf{B} \times i.S.\mathbf{B} \\ \downarrow 1 \times \amalg & & \downarrow \amalg \\ i.S.\mathbf{B} \times i.S.\mathbf{B} & \xrightarrow{\amalg} & i.S.\mathbf{B} \end{array} \quad \begin{array}{c} \nearrow \alpha \\ \nwarrow \end{array}$$

inducing a homotopy between them after realization. It would also follow that the two maps $i.S.\mathbf{B} \longrightarrow i.S.\mathbf{B}$ given by $B \longmapsto B \amalg 0$ and $B \longmapsto 0 \amalg B$ are homotopic to the identity map, hence $|i.S.\mathbf{B}|$ is unital. It seems that we do not have enough data to produce such choices in general. We shall refer to this case as *pathological*. The latter term is caused by the observation that one always has the required choices in practice. Indeed, all left systems of diagram categories or left pointed d  rivateurs arise in practice as the hyperfunctor

$$I \longmapsto \mathrm{Ho} \mathcal{C}^I$$

with \mathcal{C} being closed under coproducts. Then the choices are made in \mathcal{C} .

CONVENTION. In the rest of this section we assume \mathbf{B} to be non-pathological.

By a *right exact sequence* $F' \longrightarrow F \longrightarrow F''$ of right exact functors $\mathbf{B}' \longrightarrow \mathbf{B}$ is meant a right exact functor $G : \mathbf{B}' \longrightarrow \mathbf{E} = \mathbf{E}(\mathbf{B}, \mathbf{B}, \mathbf{B})$ such that $F' = s \circ G$, $F = t \circ G$, and $F'' = q \circ G$.

PROPOSITION 6.4 (Equivalent formulations of the additivity theorem). *Each of the following conditions implies the three others.*

(1) *The following projection*

$$i.S.\mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'') \longrightarrow i.S.\mathbf{B}' \times i.S.\mathbf{B}'', \quad E \longmapsto (E_{(0,0)}, E_{(1,1)})$$

is a homotopy equivalence.

(2) *The following projection*

$$i.S.\mathbf{E} \longrightarrow i.S.\mathbf{B} \times i.S.\mathbf{B}, \quad E \longmapsto (E_{(0,0)}, E_{(1,1)})$$

is a homotopy equivalence.

(3) *The following two maps are homotopic,*

$$i.S.\mathbf{E} \longrightarrow i.S.\mathbf{B}, \quad E \longmapsto E_{(0,1)}, \text{ respectively } E \longmapsto E_{(0,0)} \coprod E_{(1,1)}.$$

(4) *If $F' \longrightarrow F \longrightarrow F''$ is a right exact sequence of right exact functors $\mathbf{B}' \longrightarrow \mathbf{B}$ then there exists a homotopy*

$$|i.S.F| \simeq |i.S.F'| \vee |i.S.F''|.$$

Proof. (2) is a special case of (1), (3) is a special case of (4), and (4) follows from (3) by naturality.

So it will suffice to show the implications (2) \implies (3) and (4) \implies (1).

(2) \implies (3). The desired homotopy $|i.S.t| \simeq |i.S.(s \vee q)|$ is valid upon the restriction along the map

$$|i.S.\mathbf{B}| \times |i.S.\mathbf{B}| \longrightarrow |i.S.\mathbf{E}|, \quad (A, B) \longmapsto \alpha A \coprod \beta B,$$

so it will suffice to know that this map is a homotopy equivalence. But this map is a section to the map in (2) and therefore is a homotopy equivalence if that is one.

(4) \implies (1). First consider the maps $l : \Delta^1 \longrightarrow \square$, $\varkappa \longmapsto (\varkappa, 0)$, and $q : \square \longrightarrow \Delta^1$, $(\varepsilon, \eta) \longmapsto \varepsilon$. Denote by $\mathbf{E}'_? = \{q^*l^*E \mid E \in \mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'')_?\}$. Given $E \in \mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'')_0$ the object q^*l^*E is depicted in \mathbf{B}_0 as

$$\begin{array}{ccc} E_{(0,0)} & \xrightarrow{1} & E_{(0,0)} \\ \downarrow & & \downarrow \\ O & \xrightarrow{1} & O \end{array}$$

where $O = E_{(1,0)}$ is a zero object.

Also, let $i : \Delta^1 \longrightarrow \square$, $\varkappa \longmapsto (1, \varkappa)$, and $j : \square \longrightarrow \Delta^1$, $(\varepsilon, \eta) \longmapsto \eta$. Denote by $\mathbf{E}''_? = \{j^*i^*E \mid E \in \mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'')_?\}$. Given $E \in \mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'')_0$ the object j^*i^*E is depicted in \mathbf{B}_0 as

$$\begin{array}{ccc} O & \longrightarrow & E_{(1,1)} \\ 1 \downarrow & & \downarrow 1 \\ O & \longrightarrow & E_{(1,1)} \end{array}$$

We shall construct a right exact morphism

$$E \in \mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'')_? \longmapsto E^2 \in \mathbf{E}(\mathbf{E}', \mathbf{E}, \mathbf{E}'')_?$$

such that E^2 is depicted in \mathbf{E}_0 as follows.

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ O & \longrightarrow & E'' \end{array}$$

If we neglect the object at $(1, 0) \in \square$ the above diagram is depicted in \mathbf{B}_0 as

$$\begin{array}{ccccc} E_{(0,0)} & \xrightarrow{1} & E_{(0,0)} & \longrightarrow & O \\ \downarrow 1 & & \downarrow & & \downarrow \\ E_{(0,0)} & \longrightarrow & E_{(0,1)} & \longrightarrow & E_{(1,1)} \\ \downarrow & & \downarrow & & \downarrow 1 \\ O & \longrightarrow & E_{(1,1)} & \longrightarrow & E_{(1,1)}. \end{array}$$

Then it will immediately follow from assumption that

$$i.S.\mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'') \longrightarrow i.S.\mathbf{E}' \times i.S.\mathbf{E}'', \quad E \longmapsto (q^*l^*E, j^*i^*E)$$

is a homotopy equivalence with section given by $(E', E'') \longmapsto E' \amalg E''$. We construct the square E^2 in the following way. Consider two maps $\varphi, \psi : \square \longrightarrow \Delta^1$ defined by

$$(0, 0), (0, 1), (1, 0) \xrightarrow{\varphi} 0, \quad (1, 1) \xrightarrow{\varphi} 1$$

and

$$(0, 0) \xrightarrow{\psi} 0, \quad (0, 1), (1, 0), (1, 1) \xrightarrow{\psi} 1.$$

Given an object $A \in \mathbf{B}_{\Delta^1}$ the functors φ^* and ψ^* take A to the squares which are depicted in \mathbf{B}_0 as

$$\begin{array}{ccc} A_0 & \xrightarrow{1} & A_0 \\ \downarrow 1 & & \downarrow \\ A_0 & \longrightarrow & A_1 \end{array}$$

and

$$\begin{array}{ccc} A_0 & \longrightarrow & A_1 \\ \downarrow & & \downarrow 1 \\ A_1 & \xrightarrow{1} & A_1 \end{array}$$

respectively.

An object $E \in \mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'')_0$ can be regarded as an object in $\mathbf{B}(\Delta^1)_{\Delta^1}$ and it is evaluated as $E_{(0,0)} \longrightarrow O$ at 0 and as $E_{(0,1)} \longrightarrow E_{(1,1)}$ at 1. We embed the E into the cocartesian square $E^1 = (1_{\Delta^1} \times \varphi)^*E$ in $\mathbf{B}(\Delta^1)_{\square}$. After depicting E^1 in an

appropriate way, one obtains the following diagram in \mathbf{B}_0 .

$$\begin{array}{ccccc}
& & E_{(0,0)} & \longrightarrow & O \\
& \nearrow & \downarrow & & \nearrow \\
E_{(0,0)} & & O & & O \\
& \downarrow & \downarrow & & \downarrow \\
& E_{(0,1)} & \longrightarrow & E_{(1,1)} \\
& \nearrow & \downarrow & & \nearrow \\
E_{(0,0)} & & O & &
\end{array}$$

The object E^1 can be regarded as an object in $\mathbf{B}(\square)_{\Delta^1}$ and it is evaluated as the left square of the depicted cube at 0 and as the right square at 1. We embed the E^1 into the cocartesian square $E^2 = (\psi \times 1_{\square})^* E^1$ in $\mathbf{B}(\square)_{\square}$. The construction of E^2 is completed. The morphism $E \in \mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'')_{\mathbf{?}} \mapsto E^2 \in \mathbf{E}(\mathbf{E}', \mathbf{E}, \mathbf{E}'')_{\mathbf{?}}$ is induced by $(\psi \times 1_{\square})^*(1_{\Delta^1} \times \varphi)^*$.

It remains to show that the maps $f : E \in \mathbf{E}'_{\mathbf{?}} \mapsto E_{(0,0)} \in \mathbf{B}_{\mathbf{?}}$ and $g : E \in \mathbf{E}''_{\mathbf{?}} \mapsto E_{(1,1)} \in \mathbf{B}_{\mathbf{?}}$ induce a homotopy equivalence

$$i.S.\mathbf{E}' \times i.S.\mathbf{E}'' \xrightarrow{(f,g)} i.S.\mathbf{B}' \times i.S.\mathbf{B}''.$$

For the map $p : i.S.\mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'') \longrightarrow i.S.\mathbf{B}' \times i.S.\mathbf{B}''$, $E \mapsto (E_{(0,0)}, E_{(1,1)})$, equals to the composition

$$i.S.\mathbf{E}(\mathbf{B}', \mathbf{B}, \mathbf{B}'') \xrightarrow{(q^*l^*, j^*i^*)} i.S.\mathbf{E}' \times i.S.\mathbf{E}'' \xrightarrow{(f,g)} i.S.\mathbf{B}' \times i.S.\mathbf{B}''$$

and the left arrow is a homotopy equivalence by above.

Let $\bar{\mathbf{B}}_{\mathbf{?}} = \{X \in \mathbf{B}_{\Delta^1 \times \mathbf{?}} \mid X_1 \simeq 0\}$. Then $\bar{\mathbf{B}}$ and \mathbf{E}' are isomorphic because $l^*q^* = 1_{\bar{\mathbf{B}}}$ and $q^*l^*q^*l^*|_{\mathbf{E}'} = 1_{\mathbf{E}'}$. In a similar way, let $\bar{\mathbf{B}}_{\mathbf{?}} = \{X \in \mathbf{B}_{\Delta^1 \times \mathbf{?}} \mid X_0 \simeq 0\}$. Then $\bar{\mathbf{B}}$ and \mathbf{E}'' are isomorphic because $i^*j^* = 1_{\bar{\mathbf{B}}}$ and $j^*i^*j^*i^*|_{\mathbf{E}''} = 1_{\mathbf{E}''}$.

Finally, the proof of Proposition 3.1 shows that the morphism $\bar{\mathbf{B}} \longrightarrow \mathbf{B}'$ induced by the map $0 \mapsto 0 \in \Delta^1$ is an equivalence as well as the morphism $\bar{\mathbf{B}} \longrightarrow \mathbf{B}''$ induced by the map $0 \mapsto 1 \in \Delta^1$. We are done. \square

Given a simplicial object X let $PX \longrightarrow X$ be the projection induced by the face map $\partial_0 : X_{n+1} \longrightarrow X_n$. If we consider X_1 as a trivial simplicial object, there is an inclusion $X_1 \longrightarrow PX$ resulting a sequence $X_1 \longrightarrow PX \longrightarrow X$.

In particular we obtain a sequence $i.S_1\mathbf{B} \longrightarrow P(i.S.\mathbf{B}) \longrightarrow i.S.\mathbf{B}$ which in view of the equivalence of $i.S_1\mathbf{B}$ with $i.\mathbf{B}_0$ may be rewritten as

$$i.\mathbf{B}_0 \xrightarrow{G} P(i.S.\mathbf{B}) \xrightarrow{\partial_0} i.S.\mathbf{B}.$$

We show explicitly what the map G is. Let $\ell^* : S_1\mathbf{B} \longrightarrow \mathbf{B}_0$ be the equivalence stated in Proposition 3.1. A quasi-inverse to ℓ^* is constructed as follows. Consider the open immersion $e : 0 \mapsto 0 \in \Delta^1$. If \mathbf{B} is a left system of diagram categories and $k : \Delta^1 \longrightarrow 0$ is the map $0; 1 \mapsto 0; \star$, then $(k^*B)_0 = B$ and $(k^*B)_1 = 0$ for any $B \in \mathbf{B}_0$. In turn, if \mathbf{B} is a left pointed d rivateur, then $(e_*B)_0 = B$ and $(e_*B)_1 = 0$.

Next, let $p : \Delta^1 \longrightarrow \text{Ar } \Delta^1$ be the closed immersion $i \mapsto (i, 1)$, $r : \text{Ar } \Delta^{1\star} \longrightarrow \Delta^{1\star}$ be the map $(0, 0) \mapsto \star$, $(0; 1, 1) \mapsto 0; 1$, and $B \in \mathbf{B}_0$. Then $(r^*k^*B)_{(0,0)} = (r^*k^*B)_{(1,1)} = 0$ and $(r^*k^*B)_{(0,1)} = B$ if \mathbf{B} is a left system of diagram categories

and denote $g = r^*k^*$. If \mathbf{B} is a left pointed dérivateur, it follows that $(p!e_*B)_{(0,0)} = (p!e_*B)_{(1,1)} = 0$ and $(p!e_*B)_{(0,1)} = B$. In this case $g = p!e_*$.

Consider further the map $v : \Delta^{n+1} \longrightarrow \Delta^1$, $0 \longmapsto 0$ and $i \longmapsto 1$ for $i \geq 1$. We put $G = v^*g : \mathbf{B}_0 \longrightarrow S_{n+1}\mathbf{B}$. Then the “values” of GB at each $(i, j) \in \text{Ar } \Delta^{n+1}$ are: $GB_{(i,j)} = 0$ if $(i, j) = (0, 0)$ and if $i \geq 1$ and $GB_{(0,j)} = B$ for any $j \geq 1$. Considering \mathbf{B}_0 as a trivial simplicial category, there results the map $G : \mathbf{B}_0 \longrightarrow PS.\mathbf{B}$ as well as the map $G : |i.\mathbf{B}_0| \longrightarrow |P(i.S.\mathbf{B})|$.

The composite map $|i.\mathbf{B}_0| \xrightarrow{G} |P(i.S.\mathbf{B})| \longrightarrow |i.S.\mathbf{B}|$ is constant, and $|P(i.S.\mathbf{B})|$ is contractible (for it is homotopy equivalent to the contractible space $|i.S_0\mathbf{B}|$), so we obtain a map, well defined up to homotopy,

$$|i.\mathbf{B}_0| \longrightarrow \Omega|i.S.\mathbf{B}|.$$

We make a couple of useful observations due to Waldhausen [26, p. 332].

OBSERVATION. *The following two composite maps are homotopic,*

$$|i.\mathbf{E}_0| \xrightleftharpoons[s \vee q]{t} |i.\mathbf{B}_0| \longrightarrow \Omega|i.S.\mathbf{B}|.$$

Proof. This results from an inspection of $|i.S.\mathbf{B}|_{(2)}$, the 2-skeleton of $|i.S.\mathbf{B}|$ in the S -direction. We can identify $i\mathbf{B}_0$ to $iS_1\mathbf{B}$ and $i\mathbf{E}_0$ to $iS_2\mathbf{B}$.

The face maps from $i.S_2\mathbf{B}$ to $i.S_1\mathbf{B}$ then correspond to the three maps s, t, q , respectively, and each of which can be seen from the diagram

$$\begin{array}{ccc} 0 & \xrightarrow{A_{0,2}} & 2 \\ & \searrow A_{0,1} & \nearrow A_{1,2} \\ & 1 & \end{array}$$

Let us consider the canonical map $|i.S_2\mathbf{B}| \times |\Delta^2| \longrightarrow |i.S.\mathbf{B}|_{(2)}$. Regarding the 2-simplex $|\Delta^2|$ as a homotopy from the edge $(0, 2)$ to the edge path $(0, 1)(1, 2)$ we obtain a homotopy from the composite map jt ,

$$|i.\mathbf{E}_0| \xrightarrow{t} |i.\mathbf{B}_0| \xrightarrow{j} \Omega|i.S.\mathbf{B}|_{(2)}$$

to the loop product of two composite maps js and jq . But in $\Omega|i.S.\mathbf{B}|$ the loop product is homotopic to the composition law, by a well known fact about loop spaces of H -spaces, whence the observation as stated. \square

The same consideration shows, more generally,

OBSERVATION. *For every $n \geq 0$ the two composite maps*

$$|i.S.^n\mathbf{E}| \xrightleftharpoons[s \vee q]{t} |i.S.^n\mathbf{B}| \longrightarrow \Omega|i.S.^{n+1}\mathbf{B}|.$$

are homotopic.

THEOREM 6.5. *The additivity theorem (Proposition 6.4) is valid if the definition of K -theory as $\Omega|i.S.\mathbf{B}|$ is substituted with $\Omega^\infty|i.S.^\infty\mathbf{B}| = \lim_n \Omega^n|i.S.^n\mathbf{B}|$.*

Proof. By the preceding observation the two composite maps

$$\Omega^\infty|i.S.^\infty\mathbf{E}| \xrightarrow[s \vee q]{t} \Omega^\infty|i.S.^\infty\mathbf{B}| \longrightarrow \Omega^\infty|i.S.^\infty\mathbf{B}|$$

are homotopic. Since the map on the right is an isomorphism, this is one of the equivalent conditions of the additivity theorem (Proposition 6.4). \square

REMARK. As a consequence of the theorem we could add yet another reformulation of the additivity theorem to the list of Proposition 6.4 (see also Theorem 6.6). Namely the additivity theorem as stated there implies that the maps $|i.S.^n\mathbf{B}| \longrightarrow \Omega|i.S.^{n+1}\mathbf{B}|$ are homotopy equivalences for $n \geq 1$. Conversely if these maps are homotopy equivalences then so is $\Omega|i.S.\mathbf{B}| \longrightarrow \Omega^\infty|i.S.^\infty\mathbf{B}|$, and thus the additivity theorem is provided by the theorem.

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a right exact functor between two left systems of diagram categories or between two left pointed d rivateurs respectively. We denote by $\mathbf{S}.(F : \mathbf{A} \longrightarrow \mathbf{B})$ the fibred product of the diagram

$$\mathbf{S}.\mathbf{A} \xrightarrow{F} \mathbf{S}.\mathbf{B} \xleftarrow{\partial_0} PS.\mathbf{B}$$

with $\partial_0 = d_0^*$ the map induced by $d_0 : \Delta^n \longrightarrow \Delta^{n+1}$. By Proposition 1.3 $\mathbf{S}.(F : \mathbf{A} \longrightarrow \mathbf{B})$ is a simplicial left system of diagram categories or a simplicial left pointed d rivateur respectively. Thus for every n one has a commutative diagram

$$\begin{array}{ccc} \mathbf{S}_n(F : \mathbf{A} \longrightarrow \mathbf{B}) & \xrightarrow{F'} & (PS.\mathbf{B})_n = \mathbf{S}_{n+1}\mathbf{B} \\ p \downarrow & & \downarrow \partial_0 \\ \mathbf{S}_n\mathbf{A} & \xrightarrow{F} & \mathbf{S}_n\mathbf{B}. \end{array}$$

By construction we can identify an object of $\mathbf{S}_n(F : \mathbf{A} \longrightarrow \mathbf{B})_?$ to a triple (A, c, B) with objects in $\mathbf{S}_n\mathbf{A}_?$ and $\mathbf{S}_{n+1}\mathbf{B}_?$, respectively, together with an isomorphism $FA \xrightarrow{c} \partial_0 B$. We note that all the maps in the defined diagram are right exact.

Let $G : \mathbf{B} \longrightarrow \mathbf{S}_{n+1}\mathbf{B}$ be the morphism constructed above. We have $\partial_0 GB = 0$. Then G factors as $F' \circ G'$ with $G' : \mathbf{B} \longrightarrow \mathbf{S}_n(F : \mathbf{A} \longrightarrow \mathbf{B})$, $B \xrightarrow{G'} (0, 1, GB)$.

Regarding \mathbf{B} as a simplicial object in a trivial way, we obtain a sequence

$$(9) \quad \mathbf{B} \xrightarrow{G'} \mathbf{S}.(F : \mathbf{A} \longrightarrow \mathbf{B}) \xrightarrow{p} \mathbf{S}.\mathbf{A}$$

in which the composed map is trivial. There results a sequence

$$i.S.\mathbf{B} \longrightarrow i.S.S.(\mathbf{A} \longrightarrow \mathbf{B}) \longrightarrow i.S.S.\mathbf{A},$$

induced by (9).

Similarly, there is a sequence

$$i.S.\mathbf{B} \longrightarrow P(i.S.S.\mathbf{B}) \longrightarrow i.S.S.\mathbf{B}$$

where the “ P ” refers to the first S .-direction, say.

THEOREM 6.6. *The following statements are equivalent:*

- (1) *the additivity theorem (Proposition 6.4) is valid;*

(2) the sequence

$$i.S.\mathbf{B} \longrightarrow i.S.S.(\mathbf{A} \longrightarrow \mathbf{B}) \longrightarrow i.S.S.\mathbf{A},$$

is a fibration up to homotopy;

(3) the sequence

$$i.S.\mathbf{B} \longrightarrow P(i.S.S.\mathbf{B}) \longrightarrow i.S.S.\mathbf{B}$$

is a fibration up to homotopy;

(4) the map $|i.S.^n\mathbf{B}| \longrightarrow \Omega|i.S.^{n+1}\mathbf{B}|$ is a homotopy equivalence for any $n \geq 1$.

If the equivalent conditions (1) – (4) hold, then the spectrum

$$n \longmapsto i.S.^n\mathbf{B}$$

with structural maps being defined as the map $|i.\mathbf{B}_0| \longrightarrow \Omega|i.S.\mathbf{B}|$ above is a Ω -spectrum beyond the first term. The spectrum is connective (the n th term is $(n-1)$ -connected). As a consequence, the K -theory for \mathbf{B} can then equivalently be defined as the space

$$\Omega^\infty|i.S.^\infty\mathbf{B}| = \lim_n \Omega^n|i.S.^n\mathbf{B}|.$$

Proof. (3) is a consequence of (2). Since the space $|P(i.S.S.\mathbf{B})|$ is contractible, the condition (3) implies that the map $|i.S.\mathbf{B}| \longrightarrow \Omega|i.S.S.\mathbf{B}|$ is a homotopy equivalence and more generally therefore also the map $|i.S.^n\mathbf{B}| \longrightarrow \Omega|i.S.^{n+1}\mathbf{B}|$ for any $n \geq 1$. So (4) is a consequence of (3). By the second observation following Proposition 6.4 the two composite maps

$$|i.S.\mathbf{E}| \xrightarrow[s \vee q]{t} |i.S.\mathbf{B}| \longrightarrow \Omega|i.S.S.\mathbf{B}|$$

are homotopic. If the map on the right is a homotopy equivalence then t is homotopic to $s \vee q$. Thus (4) implies (1). It remains therefore to prove (1) \implies (2).

By Lemma 4.2 it is enough to show that for every n the sequence $i.S.\mathbf{B} \longrightarrow i.S.S_n(\mathbf{A} \longrightarrow \mathbf{B}) \longrightarrow i.S.S_n\mathbf{A}$ is a fibration (since the base term $i.S.S_n\mathbf{A}$ is connected for every n). Using the additivity theorem we will show that the sequence is the same, up to homotopy, as the trivial fibration sequence associated to the product $i.S.\mathbf{B} \times i.S.S_n\mathbf{A}$.

Consider the maps $u : \Delta^1 \longrightarrow \Delta^{n+1}$, $0; 1 \longmapsto 0; 1$, and $v : \Delta^{n+1} \longrightarrow \Delta^1$, $0 \longmapsto 0$, $i \longmapsto 1$ for $i \geq 1$. Then u is left adjoint to v . To simplify the notation the corresponding maps $\text{Ar } \Delta^1 \longrightarrow \text{Ar } \Delta^{n+1}$ and $\text{Ar } \Delta^{n+1} \longrightarrow \text{Ar } \Delta^1$ induced by u and v denote by the same letters. Let $\bar{\mathbf{B}} = \{v^*u^*B \mid B \in \mathbf{S}_{n+1}\mathbf{B}\}$. It follows that $v^*u^*B_{(0,0)} = B_{(0,0)}$, $v^*u^*B_{(0,i)} = B_{(0,1)}$ for any $1 \leq i \leq n+1$, and $v^*u^*B_{(i,j)} = B_{(1,1)}$ for any $i \geq 1$.

Denote by $\bar{\bar{\mathbf{B}}} = \{\sigma_0\partial_0 B \mid B \in \mathbf{S}_{n+1}\mathbf{B}\}$, where $\sigma_0 : \mathbf{S}_n\mathbf{B} \longrightarrow \mathbf{S}_{n+1}\mathbf{B}$ is the map induced by $s_0 : \text{Ar } \Delta^{n+1} \longrightarrow \text{Ar } \Delta^n$. Note that σ_0 is right adjoint to ∂_0 .

Let $m : \Delta^1 \times \text{Ar } \Delta^{n+1} \longrightarrow \text{Ar } \Delta^{n+1}$ be the map taking $(0, (i, j))$ to $(uv(i), uv(j))$ and $(1, (i, j))$ to (i, j) . Then m^* takes an object $B \in \mathbf{S}_{n+1}\mathbf{B}_?$ to that in $\mathbf{S}_{n+1}\mathbf{B}_{\Delta^1 \times ?}$, which is depicted in $\mathbf{S}_{n+1}\mathbf{B}_?$ as the adjunction morphism $v^*u^*B \longrightarrow B$.

Next, let $l : \Delta^1 \times \text{Ar } \Delta^{n+1} \longrightarrow \text{Ar } \Delta^{n+1}$ be the map taking $(0, (i, j))$ to (i, j) and $(1, (i, j))$ to $(d_0s_0(i), d_0s_0(j))$. Then l^* takes an object $B \in \mathbf{S}_{n+1}\mathbf{B}_?$ to the object in

$\mathbf{S}_{n+1}\mathbf{B}_{\Delta^1 \times ?}$, which is evaluated as B at 0, and as $\sigma_0 \partial_0 B$ at 1. This object is depicted in $\mathbf{S}_{n+1}\mathbf{B}_?$ as the adjunction morphism $\beta : B \longrightarrow \sigma_0 \partial_0 B$.

Restriction of the right exact morphism $(1_{\Delta^1} \times m)^* l^* : \mathbf{B}(\text{Ar } \Delta^{n+1}) \longrightarrow \mathbf{B}(\square \times \Delta^{n+1})$ to $\mathbf{S}_{n+1}\mathbf{B}$ takes an object $B \in \mathbf{S}_{n+1}\mathbf{B}_?$ to one in $\mathbf{E}(\text{Ar } \Delta^{n+1})_? \subset \mathbf{B}(\square \times \text{Ar } \Delta^{n+1})_?$. Thus we result in a right exact functor

$$T : \mathbf{S}_{n+1}\mathbf{B} \xrightarrow{l^*} \mathbf{S}_{n+1}\mathbf{B}(\Delta^1) \xrightarrow{(1_{\Delta^1} \times m)^*} \mathbf{E}(\text{Ar } \Delta^{n+1})$$

such that $t \circ T$ is the identity morphism on $\mathbf{S}_{n+1}\mathbf{B}$ and for every $B \in \mathbf{S}_{n+1}\mathbf{B}$ we have $s \circ T(B) \in \bar{\mathbf{B}}$ and $q \circ T(B) \in \bar{\mathbf{B}}$. So T takes its values in $\mathbf{E}(\bar{\mathbf{B}}, \mathbf{S}_{n+1}\mathbf{B}, \bar{\mathbf{B}})$ and it is actually a functor

$$T : \mathbf{S}_{n+1}\mathbf{B} \longrightarrow \mathbf{E}(\bar{\mathbf{B}}, \mathbf{S}_{n+1}\mathbf{B}, \bar{\mathbf{B}}).$$

To illustrate the reader the above procedure, let us think of $\mathbf{S}_{n+1}\mathbf{B}$ for a short while as “strings” $\mathbf{B}(\Delta^n)$ via the equivalence $\ell^* : \mathbf{S}_{n+1}\mathbf{B} \longrightarrow \mathbf{B}(\Delta^n)$ stated in Proposition 3.1. Let us consider the function $\tilde{m} : \Delta^1 \times \Delta^n \longrightarrow \Delta^n$ defined by

$$(0, i) \longmapsto 0, \quad (1, i) \longmapsto i.$$

Then the induced right exact morphism $\tilde{m}^* : \mathbf{B}(\Delta^n) \longrightarrow \mathbf{B}(\Delta^1 \times \Delta^n)$ takes an object $B \in \mathbf{B}(\Delta^n)_?$ to the object in $\mathbf{B}(\Delta^1 \times \Delta^n)_?$ which is depicted in $\mathbf{B}_?$ as

$$\begin{array}{ccccccc} B_0 & \xrightarrow{1} & B_0 & \xrightarrow{1} & \cdots & \xrightarrow{1} & B_0 \\ 1 \downarrow & & \downarrow b_1 & & & & \downarrow b_n \cdots b_1 \\ B_0 & \xrightarrow{b_1} & B_1 & \xrightarrow{b_2} & \cdots & \xrightarrow{b_n} & B_n \end{array}$$

Let $k : \Delta^1 \times \Delta^n \longrightarrow \text{Ar } \Delta^{n+1}$ be the map $(i, j) \longmapsto (i, j+1)$, $\alpha_j : \square \longrightarrow \Delta^1 \times \Delta^n$ the map taking $(0; 1, 0)$ to $(0; 1, j)$ and $(0; 1, 1)$ to $(0; 1, j+1)$. Denote by $\mathbf{S}'_{n+1}\mathbf{B}$ the left subsystem of diagram categories or the left pointed subdérivateur of $\mathbf{B}(\Delta^1 \times \Delta^n)$ respectively consisting of the objects B such that all the squares $\alpha_j^* B$, $j \leq n$, are cocartesian and $B_{(1,0)} = O$ is a zero object. Then the restriction morphisms $k^* : \mathbf{S}_{n+1}\mathbf{B} \longrightarrow \mathbf{S}'_{n+1}\mathbf{B}$ and $w^* : \mathbf{S}'_{n+1}\mathbf{B} \longrightarrow \mathbf{B}(\Delta^n)$ with $w : \Delta^n \longrightarrow \Delta^1 \times \Delta^n$, $j \longmapsto (0, j)$ are equivalences as one easily shows.

Restriction of the morphism $(1_{\Delta^1} \times \tilde{m})^* : \mathbf{B}(\Delta^1 \times \Delta^n) \longrightarrow \mathbf{B}(\square \times \Delta^n)$ to $\mathbf{S}'_n\mathbf{B}$ takes an object $B \in \mathbf{S}'_n\mathbf{B}_?$ to one in $\mathbf{B}(\square \times \Delta^n)_?$ which is depicted in $\mathbf{B}_?$ as

$$\begin{array}{ccccccc} & B_{(0,0)} & \longrightarrow & B_{(0,1)} & \longrightarrow & B_{(0,2)} & \longrightarrow \cdots \longrightarrow B_{(0,n)} \\ B_{(0,0)} & \nearrow \downarrow & B_{(0,0)} & \nearrow \downarrow & B_{(0,0)} & \nearrow \downarrow & \cdots \longrightarrow B_{(0,0)} \downarrow \\ \downarrow & \nearrow & O & \nearrow & B_{(1,1)} & \nearrow & B_{(1,2)} \longrightarrow \cdots \longrightarrow B_{(1,n)} \\ & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\ O & \longrightarrow & O & \longrightarrow & O & \longrightarrow & \cdots \longrightarrow O \end{array}$$

The back wall of the diagram is the element B of $\mathbf{S}'_n\mathbf{B}_?$ drawn in $\mathbf{B}_?$. We obtain a right exact morphism $\tilde{T} = (1_{\Delta^1} \times \tilde{m})^* \circ w^{*-1} : \mathbf{B}(\Delta^n) \longrightarrow \mathbf{E}(\Delta^n)$. For each transverse x th square $B_{x,\square}$, $x \leq n$, is cocartesian.

Now, return to the morphism T and lift it to $\mathbf{S}_n(\mathbf{A} \longrightarrow \mathbf{B})$. To be more precise, we send an object $(A, c, B) \in \mathbf{S}_n(\mathbf{A} \longrightarrow \mathbf{B})_?$ to $(\partial_0 T \sigma_0 A, \partial_0 T \sigma_0(c), TB) \in \mathbf{E}(\mathbf{S}_n(\mathbf{A} \longrightarrow \mathbf{B}))_?$. We use the commutative diagram

$$\begin{array}{ccccccc} \mathbf{A}(\mathrm{Ar} \Delta^n) & \xrightarrow{\sigma_0} & \mathbf{A}(\mathrm{Ar} \Delta^{n+1}) & \xrightarrow{T} & \mathbf{A}(\square \times \mathrm{Ar} \Delta^{n+1}) & \xrightarrow{\partial_0} & \mathbf{A}(\square \times \mathrm{Ar} \Delta^n) \\ F \downarrow & & F \downarrow & & F \downarrow & & F \downarrow \\ \mathbf{B}(\mathrm{Ar} \Delta^n) & \xrightarrow{\sigma_0} & \mathbf{B}(\mathrm{Ar} \Delta^{n+1}) & \xrightarrow{T} & \mathbf{B}(\square \times \mathrm{Ar} \Delta^{n+1}) & \xrightarrow{\partial_0} & \mathbf{B}(\square \times \mathrm{Ar} \Delta^n) \end{array}$$

to show that $F \partial_0 T \sigma_0 A = \partial_0 T \sigma_0 F A$. The relation $\partial_0 T B = \partial_0 T \sigma_0 \partial_0 B$ is straightforward.

Let $\mathbf{B}' = \{(\partial_0 v^* u^* \sigma_0 A, \partial_0 v^* u^* \sigma_0(c), v^* u^* B) \mid (A, c, B) \in \mathbf{S}_n(\mathbf{A} \longrightarrow \mathbf{B})\}$ and $\mathbf{B}'' = \{(A, c, \sigma_0 \partial_0 B) \mid (A, c, B) \in \mathbf{S}_n(\mathbf{A} \longrightarrow \mathbf{B})\}$.

There results a right exact functor

$$T' : \mathbf{S}_n(\mathbf{A} \longrightarrow \mathbf{B}) \longrightarrow \mathbf{E}(\mathbf{B}', \mathbf{S}_n(\mathbf{A} \longrightarrow \mathbf{B}), \mathbf{B}'')$$

with $s \circ T'$ sending (A, c, B) to $(\partial_0 v^* u^* \sigma_0 A, \partial_0 v^* u^* \sigma_0(c), v^* u^* B)$, $t \circ T'$ being the identity, and $q \circ T'$ sending (A, c, B) to $(A, c, \sigma_0 \partial_0 B)$. Thus we get an exact sequence $s \circ T' \longrightarrow 1 \longrightarrow q \circ T'$. It follows from our assumption that the map

$$(s \circ T', q \circ T') : S.S_n(\mathbf{A} \longrightarrow \mathbf{B}) \longrightarrow S.\mathbf{B}' \times S.\mathbf{B}''$$

is a homotopy equivalence with a homotopy inverse induced by coproduct.

Clearly, the morphism $\mathbf{B}' \longrightarrow \mathbf{B}$ taking $(\partial_0 v^* u^* \sigma_0 A, \partial_0 v^* u^* \sigma_0(c), v^* u^* B)$ to $B_{(0,1)}$ is an equivalence. Its quasi-inverse is given by G' .

Let us show that the morphism $\delta : \mathbf{S}_n \mathbf{A} \longrightarrow \mathbf{B}''$, $A \longmapsto (A, 1, \sigma_0 F A)$, is a quasi-inverse to the restriction of p to \mathbf{B}'' . Obviously δ is faithful. Given an object $(A, c, B) \in \mathbf{B}''$ the map $(1, \sigma_0(c)) : (A, 1, \sigma_0 F A) \longrightarrow (A, c, B)$ is an isomorphism. It also follows that every map $(a, b) : \delta A \longrightarrow \delta A'$ in \mathbf{B}'' equals to $(a, \sigma_0 F a)$, and hence δ is also full. We see that δ is an equivalence.

It follows that the map

$$i.S.\mathbf{B}' \times i.S.\mathbf{B}'' \longrightarrow i.S.\mathbf{B} \times i.S.S_n \mathbf{A}$$

is a homotopy equivalence, hence is so the composite

$$i.S.S_n(\mathbf{A} \longrightarrow \mathbf{B}) \longrightarrow i.S.\mathbf{B}' \times i.S.\mathbf{B}'' \longrightarrow i.S.\mathbf{B} \times i.S.S_n \mathbf{A}.$$

This homotopy equivalence fits into the following commutative diagram

$$\begin{array}{ccccc} i.S.\mathbf{B} & \longrightarrow & i.S.S_n(\mathbf{A} \longrightarrow \mathbf{B}) & \longrightarrow & i.S.S_n \mathbf{A} \\ 1 \downarrow & & \downarrow & & \downarrow 1 \\ i.S.\mathbf{B} & \longrightarrow & i.S.\mathbf{B} \times i.S.S_n \mathbf{A} & \longrightarrow & i.S.S_n \mathbf{A} \end{array}$$

Being homotopy equivalent to the trivial fibration (the lower row of the diagram), we conclude that the upper sequence is a fibration, as was to be shown. \square

REMARK. Let \mathfrak{S} be either a class of left systems of diagram categories or left pointed dérivateurs satisfying the following two conditions:

- (1) $\mathbf{B} \in \mathfrak{S}$ implies $\mathbf{S}_n \mathbf{B} \in \mathfrak{S}$ for any n ;

(2) the map $i.S.\mathbf{E} \xrightarrow{(s,q)} i.S.\mathbf{B} \times i.S.\mathbf{B}$ is a homotopy equivalence for any $\mathbf{B} \in \mathfrak{S}$.
The proof of Theorem 6.6 then shows that the spectrum

$$n \longmapsto i.S.^n\mathbf{B}$$

is a Ω -spectrum beyond the first term, and so the K -theory for every $\mathbf{B} \in \mathfrak{S}$ can then equivalently be defined as the space

$$\Omega^\infty|i.S.^\infty\mathbf{B}| = \lim_n \Omega^n|i.S.^n\mathbf{B}|.$$

A left pointed *dérivateur* \mathbf{D} of the domain $\mathcal{O}rd$ is said to be *complicial* if there is a right exact equivalence $F : \mathbf{D}\mathcal{C} \longrightarrow \mathbf{D}$ for some complicial biWaldhausen category \mathcal{C} in the sense of Thomason and which is closed under the formation of canonical homotopy pushouts and canonical homotopy pullbacks. In this case we say that \mathbf{D} is *represented* by \mathcal{C} . That equivalence induces a homotopy equivalence of bisimplicial sets $F : i.S.\mathbf{D}\mathcal{C} \longrightarrow i.S.\mathbf{D}$.

THEOREM 6.7 ([7]). *The class of complicial dérivateurs satisfies the conditions of the remark above.*

PROPOSITION 6.8. *Under the hypotheses of Theorem 6.6 suppose we are given a sequence $\mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{C}$ of right exact morphisms between left systems of diagram categories or left pointed dérivateurs respectively. Then the square*

$$\begin{array}{ccc} i.S.\mathbf{B} & \longrightarrow & i.S.S.(\mathbf{A} \longrightarrow \mathbf{B}) \\ \downarrow & & \downarrow \\ i.S.\mathbf{C} & \longrightarrow & i.S.S.(\mathbf{A} \longrightarrow \mathbf{C}) \end{array}$$

is homotopy cartesian.

Proof. There is a commutative diagram

$$\begin{array}{ccccc} i.S.\mathbf{B} & \longrightarrow & i.S.S.(\mathbf{A} \longrightarrow \mathbf{B}) & \longrightarrow & i.S.S.\mathbf{A} \\ \downarrow & & \downarrow & & \downarrow \text{id} \\ i.S.\mathbf{C} & \longrightarrow & i.S.S.(\mathbf{A} \longrightarrow \mathbf{C}) & \longrightarrow & i.S.S.\mathbf{A} \end{array}$$

in which the rows are fibrations up to homotopy by Theorem 6.6. Therefore the square on the left is homotopy cartesian. \square

COROLLARY 6.9. *Under the hypotheses of Theorem 6.6 the following two assertions are valid.*

(1) *To a right exact morphism there is associated a fibration*

$$i.S.\mathbf{B} \longrightarrow i.S.\mathbf{C} \longrightarrow i.S.S.(\mathbf{B} \longrightarrow \mathbf{C}).$$

(2) *If \mathbf{C} is a retract of \mathbf{B} (by right exact functors) there is a splitting*

$$i.S.\mathbf{B} \simeq i.S.\mathbf{C} \times i.S.S.(\mathbf{C} \longrightarrow \mathbf{B}).$$

Proof. (1). If $\mathbf{A} = \mathbf{B}$ the space $|i.S.S.(\mathbf{A} = \mathbf{A})|$ is contractible whence the first assertion.

(2). This is the case of Proposition 6.8 where the composed map $\mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{C}$ is an identity map since $i.S.S.(\mathbf{A} \longrightarrow \mathbf{C})$ is contractible in that case. \square

7. CONCLUDING REMARKS

Given an exact category \mathcal{E} , one would like to compare Quillen's K -theory $K(\mathcal{E})$ of \mathcal{E} with the K -theory of the associated bidérivateur $\mathbf{D}^b(\mathcal{E})$.

Let $wC^b(\mathcal{E})$ denote the Waldhausen category of quasi-isomorphisms in $C^b(\mathcal{E})$ with cofibrations componentwise admissible monomorphisms. We have a natural functor for every $I \in \mathcal{D}irf$,

$$\mathrm{Ho} : C^b(\mathcal{E}^I) \longrightarrow D^b(\mathcal{E}^I).$$

The image under the functor Ho of any cocartesian square of $C^b(\mathcal{E})^\square = C^b(\mathcal{E}^\square)$

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

in which the horizontal arrows are cofibrations is a cocartesian square in $\mathbf{D}^b(\mathcal{E})^\square$ (this is dual to [4, 3.14]). Therefore Ho induces a map of bisimplicial objects

$$\nu : w.S.C^b(\mathcal{E}) \longrightarrow i.S.\mathbf{D}^b(\mathcal{E}).$$

Consider the map

$$K(\tau) : K(\mathcal{E}) \longrightarrow K(wC^b(\mathcal{E}))$$

which is induced by the map τ taking an object of \mathcal{E} to the complex concentrated in the zeroth degree ($K(wC^b(\mathcal{E}))$ stands for the Waldhausen K -theory of $wC^b(\mathcal{E})$).

QUESTION (The first Malsiniotis conjecture [14]). *The map $K(\rho) = K(\nu\tau) : K(\mathcal{E}) \longrightarrow K(\mathbf{D}^b(\mathcal{E}))$ is a homotopy equivalence.*

The particular map $K_0(\mathcal{E}) \longrightarrow K_0(\mathbf{D}^b(\mathcal{E}))$ is an isomorphism for the Grothendieck groups $K_0(\mathcal{E})$ and $K_0(D^b(\mathcal{E}))$ are naturally isomorphic (exercise!) and $K_0(\mathbf{D}^b(\mathcal{E}))$ is naturally isomorphic to $K_0(D^b(\mathcal{E}))$ by Lemma 3.5.

The first Malsiniotis conjecture is very resistant in general. However one can obtain some information for a large class of exact categories including the abelian categories. The following shows that Quillen's K -theory $K(\mathcal{E})$ of an exact category \mathcal{E} from this class is a retract of $K(\mathbf{D}^b(\mathcal{E}))$.

THEOREM 7.1. *Let \mathcal{E} be an extension closed full exact subcategory of an abelian category \mathcal{A} satisfying the conditions of the Resolution Theorem. That is*

(1) *if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is exact in \mathcal{A} and $M, M'' \in \mathcal{E}$, then $M' \in \mathcal{E}$ and*

(2) *for any object $M \in \mathcal{A}$ there is a finite resolution $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ with $P_i \in \mathcal{E}$.*

Then the map

$$K(\rho) : K(\mathcal{E}) \longrightarrow K(\mathbf{D}^b(\mathcal{E}))$$

is a split inclusion in homotopy. There is a map

$$p : K(\mathbf{D}^b(\mathcal{E})) \longrightarrow K(\mathcal{E})$$

which is left inverse to it. That is $p \circ K(\rho)$ is homotopic to the identity. In particular, each K -group $K_n(\mathcal{E})$ is a direct summand of $K_n(\mathbf{D}^b(\mathcal{E}))$.

We postpone the proof. One should remark that it essentially uses Neeman's results [15] on K -theory for triangulated categories.

It is shown in [22, 23] that the natural morphism $K(\mathcal{C}) \longrightarrow K(\mathbf{D}\mathcal{C})$ from the Waldhausen K -theory to the K -theory of its dérivateur can not be an equivalence in general. For instance this is so for the Waldhausen K -theory of spaces. This does not mean however that the K -groups $K_n(\mathcal{C})$ can not be reconstructed from its dérivateur and that this is a counter example to the comparison problem above stated for exact categories. The obstruction is really concerned with functoriality at the level of spectra.

A *good* Waldhausen category is a Waldhausen category that can be embedded in the category of cofibrant objects of a pointed model category, and whose Waldhausen structure is induced by the ambient model structure (see the precise definition in [23]). Though there exist non-good Waldhausen categories (see [23, Example 2.2]), in practice it turns out that given a Waldhausen category there is always a good Waldhausen model, i.e. a good Waldhausen category with the same K -theory space up to homotopy. Any good Waldhausen category is a Waldhausen category of cofibrant objects, and therefore one can associate to it the left pointed dérivateur $\mathbf{D}\mathcal{C}$ (Theorem 2.8). The following theorem is also a consequence of a result by Cisinski and Toën [22, 2.16].

THEOREM 7.2 (first stated by Toën [21]). *Let \mathcal{C} and \mathcal{C}' be two good Waldhausen categories such that their associated dérivateurs $\mathbf{D}\mathcal{C}$ and $\mathbf{D}\mathcal{C}'$ are equivalent. Then the Waldhausen K -theory spectra $K(\mathcal{C})$ and $K(\mathcal{C}')$ are equivalent as well.*

Under certain extra data, \mathbf{B} encodes the structure of a triangulated category on \mathbf{B}_0 [4, 6, 13, 14]. This structure is canonically carried over all the categories \mathbf{B}_I , $I \in \mathcal{D}ia$. In this case \mathbf{B} is referred to as a *system of triangulated diagram categories* or *triangulated dérivateur* respectively. The following result shows that such \mathbf{B} contains strictly more information than its triangulated category \mathbf{B}_0 .

PROPOSITION 7.3. *There exist two non-equivalent triangulated dérivateurs \mathbf{B} and \mathbf{B}' , whose associated triangulated categories \mathbf{B}_0 and \mathbf{B}'_0 are equivalent.*

Proof. Let $\mathcal{C} = m\mathcal{M}(\mathbf{Z}/p^2)$ and $\mathcal{C}' = m\mathcal{M}(\mathbf{Z}/p[\varepsilon]/\varepsilon^2)$ be two stable model categories considered in [18]. Here $\mathcal{M}(\mathbf{Z}/p^2)$ and $\mathcal{M}(\mathbf{Z}/p[\varepsilon]/\varepsilon^2)$ denote the corresponding categories of finitely generated modules. Since both \mathbf{Z}/p^2 and $\mathbf{Z}/p[\varepsilon]/\varepsilon^2$ are quasi-Frobenius rings, it follows that $\mathcal{M}(\mathbf{Z}/p^2)$ and $\mathcal{M}(\mathbf{Z}/p[\varepsilon]/\varepsilon^2)$ are Frobenius categories and $\mathbf{D}\mathcal{C}$ and $\mathbf{D}\mathcal{C}'$ are triangulated dérivateurs by [4, 4.19]. It follows from [18, 1.4] that $\mathbf{D}\mathcal{C}_0$ and $\mathbf{D}\mathcal{C}'_0$ are equivalent as triangulated categories. But the dérivateurs $\mathbf{D}\mathcal{C}$ and $\mathbf{D}\mathcal{C}'$ can not be equivalent by Theorem 7.2, because the Waldhausen K -theories $K(\mathcal{C})$ and $K(\mathcal{C}')$ are not equivalent by [18, 1.7]. \square

Another problem arising in our context (see also [14, Conjecture 2]) is the localization theorem. Suppose we are given a family $\mathcal{W} = \{\mathcal{W}_I \subseteq \text{Mor } \mathbf{B}_I \mid I \in \mathcal{D}ia\}$ of morphisms compatible with the structure functors f^* and $f_!$; that is $f^*(\mathcal{W}_J) \subseteq \mathcal{W}_I$ and $f_!(\mathcal{W}_I) \subseteq \mathcal{W}_J$ for every map $f : I \longrightarrow J$. Let $\mathbf{B}_?[\mathcal{W}_?^{-1}]$ denote the category of fractions obtained by inverting the maps in $\mathcal{W}_?$. We also require the following condition to hold: a morphism is in $\mathcal{W}_?$ iff its image in $\mathbf{B}_?[\mathcal{W}_?^{-1}]$ is an isomorphism.

Let the hyperfunctor

$$I \xrightarrow{Q} \mathbf{B}_I[\mathscr{W}_I^{-1}]$$

determine a left system of diagram categories or a left pointed dérivateur respectively. Denote it by $\mathbf{B}[\mathscr{W}^{-1}]$. Suppose further that the quotient morphism $Q : \mathbf{B} \longrightarrow \mathbf{B}[\mathscr{W}^{-1}]$ is right exact.

In the case when \mathbf{B} is a system of triangulated diagram categories or a triangulated dérivateur respectively, then any thick subcategory \mathbf{A}_0 of \mathbf{B}_0 gives rise to a localization in \mathbf{B} . Precisely, given $I \in \mathcal{D}ia$ let $\mathbf{A}_I = \{A \in \mathbf{B}_I \mid A_x \in \mathbf{A}_0 \text{ for all } x \in I\}$. Then \mathbf{A}_I is thick in \mathbf{B}_I and the functor

$$I \longmapsto \mathbf{A}_I$$

determines a system of triangulated diagram categories or a triangulated dérivateur respectively and the quotient is then naturally constructed (see [6, p. 39]).

QUESTION (The second Maltiniotis conjecture [14]). *Suppose we are given a sequence of morphisms between left systems of diagram categories or left pointed dérivateurs respectively,*

$$\mathbf{A} \xrightarrow{F} \mathbf{B} \xrightarrow{Q} \mathbf{B}[\mathscr{W}^{-1}]$$

where Q is the quotient morphism and F is a right exact equivalence between \mathbf{A} and $Q^{-1}(0) = \{B \in \mathbf{B}_? \mid 0 \longrightarrow B \in \mathscr{W}_?\}$. Then the induced sequence of K -theory spaces

$$K(\mathbf{A}) \longrightarrow K(\mathbf{B}) \longrightarrow K(\mathbf{B}[\mathscr{W}^{-1}])$$

is a fibration up to homotopy.

We have already associated to the morphism F a fibration (see Corollary 5.4(1))

$$i.S.\mathbf{A} \longrightarrow i.S.\mathbf{B} \longrightarrow i.N.S.(\mathbf{A} \longrightarrow \mathbf{B}).$$

There is a natural map from $i.N.S.(\mathbf{A} \longrightarrow \mathbf{B})$ to $i.S.\mathbf{B}[\mathscr{W}^{-1}]$. Therefore the localization theorem is reduced, say, to showing that the latter map is a homotopy equivalence.

To conclude, we should also mention another natural construction associated to a model category \mathcal{C} , the simplicial localization $L^H\mathcal{C}$, which should carry roughly the same homotopical information about \mathcal{C} as its dérivateur $\mathbf{D}\mathcal{C}$. Given a good Waldhausen category \mathcal{C} , Toën and Vezzosi [23] associate to $L^H\mathcal{C}$ a K -theory space $K(L^H\mathcal{C})$ and show that the Waldhausen K -theory $K(\mathcal{C})$ is equivalent to $K(L^H\mathcal{C})$. We also recommend the reader to consult Toën's thesis [22].

It remains to prove, as promised, Theorem 7.1. We start with preparations.

DEFINITION. An additive category \mathcal{T} will be called a *category with squares* provided

- ◇ \mathcal{T} has an automorphism $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$;
- ◇ \mathcal{T} comes equipped with a collection of *special squares*

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array} \quad \begin{array}{c} \curvearrowright \\ (1) \end{array}$$

This means that the square

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

is commutative in \mathcal{T} , and there is a map $D \longrightarrow \Sigma A$ depicted as the curly arrow. The (1) in the label of the arrow is to remind us that the map is of degree 1, that is a map $D \longrightarrow \Sigma A$.

Given two categories with squares, a *special functor*

$$F : \mathcal{S} \longrightarrow \mathcal{T}$$

is an additive functor such that there is a natural isomorphism $\Sigma F \simeq F \Sigma$ and F takes special squares in \mathcal{S} to special squares in \mathcal{T} .

If \mathcal{T} is a category with squares, the *fold of the square*

$$\begin{array}{ccc} C & \xrightarrow{\delta} & D \\ \beta \uparrow & & \uparrow \gamma \\ A & \xrightarrow{\alpha} & B \end{array} \quad \begin{array}{c} \text{curly arrow} \\ \mu \end{array}$$

will be the sequence

$$A \xrightarrow{(\alpha, -\beta)^t} B \oplus C \xrightarrow{(\gamma, \delta)} D \xrightarrow{\mu} \Sigma A.$$

EXAMPLES. Let \mathcal{T} be a triangulated category. Then \mathcal{T} is additive and comes with an automorphism Σ . A square is defined to be special iff its fold is a distinguished triangle in \mathcal{T} . When we think of a triangulated category \mathcal{T} as being the category with squares defined above, then we shall denote it as \mathcal{T}^d .

Let \mathcal{A} be an abelian category. Let $Gr^b \mathcal{A}$ be the category of bounded, graded objects in \mathcal{A} . We recall the reader that a graded object of \mathcal{A} is a sequence of objects $\{A_i \in \mathcal{A}\}_{i \in \mathbb{Z}}$. The sequence $\{A_i\}$ is bounded if $A_i = 0$ except for finitely many $i \in \mathbb{Z}$.

We define the functor $\Sigma : Gr^b \mathcal{A} \longrightarrow Gr^b \mathcal{A}$ to be the shift, that is $\Sigma\{A_i\} = \{B_i\}$ with $B_i = A_{i+1}$. A square in $Gr^b \mathcal{A}$ is defined to be special if the fold

$$A \xrightarrow{(\alpha, -\beta)^t} B \oplus C \xrightarrow{(\gamma, \delta)} D \xrightarrow{\mu} \Sigma A$$

gives a long exact sequence in \mathcal{A}

$$\cdots \longrightarrow D_{i-1} \longrightarrow A_i \longrightarrow B_i \oplus C_i \longrightarrow D_i \longrightarrow A_{i+1} \longrightarrow \cdots$$

Let $H : D^b(\mathcal{A}) \longrightarrow Gr^b \mathcal{A}$ be the homology functor taking a complex $A \in D^b(\mathcal{A})$ to $\{H_i(A)\}$. Then it induces a functor between categories with squares

$$H : D^b(\mathcal{A})^d \longrightarrow Gr^b \mathcal{A}.$$

DEFINITION. (1) Let \mathcal{T} be a category with squares and $m, n \geq 0$. A functor $X : \Delta^m \times \Delta^n \longrightarrow \mathcal{T}$ is called an *augmented diagram* if for any $0 \leq i \leq i' \leq m$ and $0 \leq j \leq j' \leq n$ we are given a special square

$$\begin{array}{ccc} X_{ij'} & \longrightarrow & X_{i'j'} \\ \uparrow & & \uparrow \\ X_{ij} & \longrightarrow & X_{i'j} \end{array} \quad \delta_{i,j}^{i',j'}$$

such that $\delta_{i,j}^{i',j'}$ is the composite

$$X_{i'j'} \longrightarrow X_{mn} \xrightarrow{\delta_{0,0}^{m,n}} \Sigma X_{00} \longrightarrow \Sigma X_{ij}.$$

By a *morphism* between augmented diagrams $\varphi : X \longrightarrow Y$ is meant a natural transformation of functors such that the square

$$\begin{array}{ccc} X_{i'j'} & \xrightarrow{\delta_{i,j}^{i',j'}} & \Sigma X_{ij} \\ \varphi_{i'j'} \downarrow & & \downarrow \Sigma \varphi_{ij} \\ Y_{i'j'} & \xrightarrow{\delta_{i,j}^{i',j'}} & \Sigma Y_{ij} \end{array}$$

is commutative for any $0 \leq i \leq i' \leq m$ and $0 \leq j \leq j' \leq n$.

The category of augmented diagrams will be denoted by $Q_{m,n}\mathcal{T}$. There results a bisimplicial category $Q\mathcal{T} = \{Q_{m,n}\mathcal{T}\}_{m,n \geq 0}$ (the face/degeneracy operators are defined by deleting/inserting a row or column).

(2) For a category with squares \mathcal{T} , its K -theory $K(\mathcal{T})$ is defined to be the space $\Omega|\text{Ob}(Q\mathcal{T})|$.

Let $H : D^b(\mathcal{A})^d \longrightarrow Gr^b \mathcal{A}$ be the functor of categories with squares constructed above. We have the map of bisimplicial categories $\chi : QD^b(\mathcal{A})^d \longrightarrow QGr^b \mathcal{A}$ induced by H , and hence the map $K(\chi) : K(D^b(\mathcal{A})) \longrightarrow K(Gr^b \mathcal{A})$.

Let \mathcal{E} be an exact category and $m, n \geq 0$. Denote by $Q_{m,n}\mathcal{E}$ the following category. Its objects are the functors $X : \Delta^m \times \Delta^n \longrightarrow \mathcal{E}$ such that for any $0 \leq i \leq i' \leq m$ and $0 \leq j \leq j' \leq n$ we are given a bicartesian square

$$\begin{array}{ccc} X_{ij} & \xrightarrow{\hookrightarrow} & X_{i'j'} \\ \uparrow & & \uparrow \\ X_{ij} & \xrightarrow{\hookrightarrow} & X_{i'j} \end{array}$$

in which the vertical arrows are epimorphisms and the horizontal arrows are monomorphisms. The morphisms are defined by natural transformations. The resulting bisimplicial category denote by $Q\mathcal{E}$. It is well-known that a simplicial model for a delooping of the space $K(\mathcal{E})$ is given by the realization of the bisimplicial set $\text{Ob } Q\mathcal{E}$.

Let \mathcal{A} be an abelian category and let $i : \mathcal{A} \longrightarrow D^b(\mathcal{A})$ denote the natural functor sending an object $A \in \mathcal{A}$ to the complex concentrated in the zeroth degree.

Then it induces a functor (see also some discussion below) of bisimplicial categories $\iota : Q\mathcal{A} \longrightarrow QD^b(\mathcal{A})^d$. Note that the differentials $\delta_{i,j}^{i',j'}$ in $QD^b(\mathcal{A})^d$ are canonically unique for every diagram coming from $Q\mathcal{A}$ (see [16]).

THEOREM 7.4 (Neeman [15]). *Let \mathcal{A} be a small abelian category. Then the composite*

$$\mathrm{Ob} Q\mathcal{A} \xrightarrow{\iota} \mathrm{Ob} QD^b(\mathcal{A})^d \xrightarrow{\chi} \mathrm{Ob} QGr^b(\mathcal{A})$$

is a homotopy equivalence.

As usual, given a category \mathcal{C} denote by $i\mathcal{C}$ the maximal groupoid in \mathcal{C} and by $i.\mathcal{C}$ the nerve in the i -direction.

COROLLARY 7.5. *Let \mathcal{A} be a small abelian category. Then the composite of maps of trisimplicial objects*

$$i.Q\mathcal{A} \xrightarrow{\iota} i.QD^b(\mathcal{A})^d \xrightarrow{\chi} i.QGr^b(\mathcal{A})$$

is a homotopy equivalence.

Proof. Given $k \geq 0$ the category $i_k\mathcal{A}$ of strings of isomorphisms $A_0 \xrightarrow{\sim} \cdots \xrightarrow{\sim} A_k$ is abelian and the composite

$$i_k Q\mathcal{A} = Q[i_k\mathcal{A}] \xrightarrow{\iota} i_k QD^b(\mathcal{A})^d \xrightarrow{\chi} i_k QGr^b(\mathcal{A}) = QGr^b[i_k\mathcal{A}]$$

is a homotopy equivalence of bisimplicial objects by Theorem 7.4. It follows from Lemma 4.1 that is so the map of the corollary. \square

Proof of Theorem 7.1. (1) First prove the statement for an abelian category \mathcal{A} . For Quillen's K -theory $K(\mathcal{A})$ we use the following simplicial model. It is the loop space of the realization of $i.Q\mathcal{A}$ (see [26]). In turn, the model for $K(\mathbf{D}^b(\mathcal{A}))$ is given by the bisimplicial maximal groupoid $i.Q\mathbf{D}^b(\mathcal{A})$ (see section 3).

By Corollary 7.5 it suffices to show that the map $i.Q\mathcal{A} \xrightarrow{\iota} i.QD^b(\mathcal{A})^d$ factors through $i.Q\mathbf{D}^b(\mathcal{A})$. Recall that $Q_{m,n}\mathbf{D}^b(\mathcal{A})$, $m, n \geq 0$, consists of the objects $X \in \mathbf{D}^b(\mathcal{A})_{\Delta^m \times \Delta^n}$ such that for any $0 \leq i \leq i' \leq m$ and $0 \leq j \leq j' \leq n$ the square

$$\begin{array}{ccc} X_{ij'} & \longrightarrow & X_{i'j'} \\ \uparrow & & \uparrow \\ X_{ij} & \longrightarrow & X_{i'j} \end{array}$$

is bicartesian in $\mathbf{D}^b(\mathcal{A})_{\square}$ (= cocartesian in triangulated d erivateurs [14]). It follows that

$$\mathrm{cone}(X_{ij} \longrightarrow X_{ij'} \oplus X_{i'j}) \longrightarrow \mathrm{cone}(0 \longrightarrow X_{i'j'}) \simeq X_{i'j'}$$

is a quasi-isomorphism in $C^b(\mathcal{A})$, hence an isomorphism in $D^b(\mathcal{A})$ (we use here properties of triangulated d erivateurs and the triangulated structure information which $\mathbf{D}^b(\mathcal{A})$ encodes [4, 6, 13]). Now compose the inverse of this isomorphism with the natural projection

$$\mathrm{cone}(X_{ij} \longrightarrow X_{ij'} \oplus X_{i'j}) \longrightarrow \mathrm{cone}(X_{ij} \longrightarrow 0) \simeq \Sigma X_{ij}$$

and we have a map $\delta_{i,j}^{i',j'} : X_{i'j'} \longrightarrow \Sigma X_{ij}$. This produces a special square in $D^b(\mathcal{A})^d$.

The construction is clearly natural. Let $f : X \longrightarrow Y$ with $X, Y \in Q_{m,n}\mathbf{D}^b(\mathcal{A})$ be an isomorphism. It is represented by a diagram

$$X \longleftarrow Z \longrightarrow Y$$

with $Z \in Q_{m,n}\mathbf{D}^b(\mathcal{A})$ and arrows quasi-isomorphisms. We have the following commutative diagram in $C^b(\mathcal{A})$ for any $0 \leq i \leq i' \leq m$ and $0 \leq j \leq j' \leq n$:

$$\begin{array}{ccccccc} X_{ij} & \xrightarrow{a} & X_{ij'} \oplus X_{i'j} & \longrightarrow & X_{i'j'} & \longleftarrow & \text{cone}(a) \longrightarrow \Sigma X_{ij} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ Z_{ij} & \xrightarrow{b} & Z_{ij'} \oplus Z_{i'j} & \longrightarrow & Z_{i'j'} & \longleftarrow & \text{cone}(b) \longrightarrow \Sigma Z_{ij} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_{ij} & \xrightarrow{c} & Y_{ij'} \oplus Y_{i'j} & \longrightarrow & Y_{i'j'} & \longleftarrow & \text{cone}(c) \longrightarrow \Sigma Y_{ij} \end{array}$$

This yields an isomorphism of triangles in $D^b(\mathcal{A})$

$$\begin{array}{ccccccc} X_{ij} & \longrightarrow & X_{ij'} \oplus X_{i'j} & \longrightarrow & X_{i'j'} & \xrightarrow{\delta_{i,j}^{i',j'}} & \Sigma X_{ij} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_{ij} & \longrightarrow & Y_{ij'} \oplus Y_{i'j} & \longrightarrow & Y_{i'j'} & \xrightarrow{\delta_{i,j}^{i',j'}} & \Sigma Y_{ij} \end{array}$$

and hence an isomorphism of special squares in $D^b(\mathcal{A})^d$.

Now, let $X \in Q_{m,n}\mathbf{D}^b(\mathcal{A})$ and $0 \leq i' \leq m$ and $0 \leq j' \leq n$. There is a commutative diagram in $C^b(\mathcal{A})$

$$\begin{array}{ccccc} X_{00} & \longrightarrow & X_{i'0} \oplus X_{0j'} & \longrightarrow & X_{i'j'} \\ \downarrow & & \downarrow & & \downarrow \\ X_{00} & \longrightarrow & X_{m0} \oplus X_{0n} & \longrightarrow & X_{mn} \end{array}$$

and hence, in $D^b(\mathcal{A})$ we deduce a commutative square

$$\begin{array}{ccc} X_{i'j'} & \xrightarrow{\delta_{0,0}^{i',j'}} & \Sigma X_{00} \\ \downarrow & & \downarrow \\ X_{mn} & \xrightarrow{\delta_{0,0}^{m,n}} & \Sigma X_{00} \end{array}$$

Given any $0 \leq i \leq i'$ and $0 \leq j \leq j'$ there is a commutative diagram in $C^b(\mathcal{A})$

$$\begin{array}{ccccc} X_{00} & \longrightarrow & X_{i'0} \oplus X_{0j'} & \longrightarrow & X_{i'j'} \\ \downarrow & & \downarrow & & \downarrow \\ X_{ij} & \longrightarrow & X_{i'j} \oplus X_{ij'} & \longrightarrow & X_{i'j'} \end{array}$$

and hence, in $D^b(\mathcal{A})$ we deduce a commutative square

$$\begin{array}{ccc} X_{i'j'} & \xrightarrow{\delta_{0,0}^{i',j'}} & \Sigma X_{00} \\ \downarrow & & \downarrow \\ X_{i'j'} & \xrightarrow{\delta_{i,j}^{i',j'}} & \Sigma X_{ij} \end{array}$$

and hence the “natural” map $\delta_{i,j}^{i',j'} : X_{i'j'} \longrightarrow \Sigma X_{ij}$ is obtained from $\delta_{0,0}^{m,n} : X_{mn} \longrightarrow \Sigma X_{00}$ just as the composite

$$X_{i'j'} \longrightarrow X_{mn} \xrightarrow{\delta_{0,0}^{m,n}} \Sigma X_{00} \longrightarrow \Sigma X_{ij}.$$

It follows that the functors $dia : \mathbf{D}^b(\mathcal{A})_{\Delta^m \times \Delta^n} \longrightarrow \text{Hom}(\Delta^m \times \Delta^n, D^b(\mathcal{A}))$, $m, n \geq 0$, induce a map of bisimplicial groupoids

$$\theta : iQ\mathbf{D}^b(\mathcal{A}) \longrightarrow iQD^b(\mathcal{A})^d.$$

Obviously, the map $i.Q\mathcal{A} \xrightarrow{\iota} i.Q\mathbf{D}^b(\mathcal{A})^d$ factors as

$$i.Q\mathcal{A} \xrightarrow{\rho} i.Q\mathbf{D}^b(\mathcal{A}) \xrightarrow{\theta} i.QD^b(\mathcal{A})^d.$$

This implies the claim.

(2) Suppose now that an exact category $\mathcal{E} \subseteq \mathcal{A}$ satisfies the assumptions of the theorem. Consider the commutative diagram

$$\begin{array}{ccc} i.Q\mathcal{E} & \xrightarrow{\rho} & i.Q\mathbf{D}^b(\mathcal{E}) \\ \downarrow & & \downarrow \\ i.Q\mathcal{A} & \xrightarrow{\rho} & i.Q\mathbf{D}^b(\mathcal{A}) \xrightarrow{\chi\theta} i.QGr^b\mathcal{A} \end{array}$$

in which the vertical arrows are induced by the inclusion $\mathcal{E} \longrightarrow \mathcal{A}$. The left vertical arrow is a homotopy equivalence by the Resolution Theorem [17]. The fact that the map $\chi\theta\rho$ is a homotopy equivalence by (1) obviously finishes the proof. \square

Basing on Vaknin’s computations [24] Neeman shows [16, p. 39] that there is an exact category \mathcal{E} such that the homomorphism $K_1(\iota) : K_1(\mathcal{E}) \longrightarrow K_1(D^b(\mathcal{E}))$ is not a monomorphism (while it is a split monomorphism for abelian categories [15, 16]). The simplest example is where \mathcal{E} is the category of free modules of finite rank over the ring of dual numbers $k[\varepsilon]/\varepsilon^2$. Such exact categories could give us counter-examples to the first Maltisiotis conjecture if we showed in a similar way that the map $K_1(\rho) : K_1(\mathcal{E}) \longrightarrow K_1(\mathbf{D}^b(\mathcal{E}))$ is not a monomorphism.

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